Parametric and Nonparametric Bootstrap in Actuarial Practice

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Abstract

Uncertainty of insurance liabilities has always been the key issue in actuarial theory and practice. This is represented for instance by study and modeling of mortality in life insurance, and loss distributions in traditional actuarial science. These models have evolved from early simple deterministic calculations to more sophisticated probabilistic ones. Such probabilistic models have been traditionally built around parameters characterizing certain probability distributions, e.g., Gompertz's model of force of mortality, or parametric models of the yield curve process.

Modern actuarial science has introduced probabilistic models for all input variables in studying insurance firm, and in the whole company models. These new methodologies are based on the theoretical work in mathematical finance which shows that the market, or fair value of insurance liabilities, and indeed, the market value of the insurance firm, can be determined using the general approach developed for contingent claims valuation. While this is theoretically appealing and justified, the central dilemma in modeling insurance company, i.e., its assets and liabilities, is the choice of an appropriate probability distributions, or stochastic processes, governing the evolution of the underlying variables such as interest rates, or asset returns in general, withdrawals or lapses, and mortality. The traditional approaches to this problem have been based on the parametric models. The last two decades have brought about a rich body of new research in nonparametric statistics. This work is intended at showing direct application of a specific nonparametric methodology in actuarial models.

The methodology researched here is that of the bootstrap, and more generally, resampling. We develop the bootstrap model alternative on both the asset and liability side. First, we show how bootstrap can be used successfully in enhancing a parametric mortality law suggested by Carriere (1992). Next, we develop a whole company asset-liability model to study a bootstrap alternative to lognormal and stable Paretian models of interest rate process. The results indicate that bootstrap can be instrumental in understanding the rich structure of random variables on the asset and liability sides of an insurance firm balance sheet, and in error estimation for the existing models.

1 Introduction

Actuarial science has always been concerned with uncertainty and the process of modeling it. In fact, the uncertainty of insurance liabilities may be considered the very reason for the creation

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of actuarial science. This is clearly represented in a study of mortality and other decrements in life, disability, and health insurance, and loss distribution in property/casualty insurance. The models used in such studies began as tabular ones, and then developed into a study of probability distributions. Probability distributions have been traditionally characterized by some numerical parameters. Examples of such probability distributions include Makeham of Gompertz laws of mortality, or the use of gamma distributions in modeling losses, or the omnipresent use of normal approximations (including the lognormal distribution modeling of interest rates).

The last two decades have brought about a new vast body of statistical research in nonparametric approaches to modeling uncertainty, in which not the individual parameters of the probability distribution, but rather the entire distribution is sought based on empirical data available. There are varying approaches in nonparametric statistics. For instance, Klugman, Panjer and Willmot (1998) give some examples of the use of kernel density estimation, or nonparametric Bayesian estimation.

One of the key new methodologies of nonparametric statistics is the concept of *the bootstrap*, also knows under a somewhat broader term of *resampling*. In this work, we present the basic ideas of the bootstrap and resampling, and show how promising its applications in actuarial modeling can be.

The paper is organized as follows. In the next section we give a brief overview of the basic ideas of the bootstrap methodology in the context of parametric and non-parametric estimation as well as time series-based inferences. The subsequent two sections of the paper are devoted to the somewhat detailed discussion of two particular examples, arising from the actuarial modeling problems. In Section 3 we we look at mortality estimation, studying a mortality law introduced by Carriere (1992), generalizing on the classical Gompertz mortality law. In Section 4, we proceed to a whole company asset-liability model, comparing cash flow testing results for an annuity company under two parametric assumptions of interest rate process and the method based on the nonparametric, bootstrap-based approach to analyzing time series data. Section 5 contains some discussion of the obtained results whereas Section **??** offers some concluding remarks.

2 The Bootstrap

Before we discuss the applications of resampling methodology, let us first give a brief overview of its basic ideas.

2.1 The Concept

The concept of the bootstrap was first introduced in the seminal piece of Efron (1979) as an attempt to give some new perspective to an old and established statistical procedure known as *jackknifing*. Unlike jackknifing which is mostly concerned with calculating standard errors of statistics of interest, Efron's bootstrap has been set to achieve more ambitious goal of estimating not only the standard error but also the distribution of a statistic. In his paper Efron has considered two types of bootstrap procedures useful, respectively, for nonparametric and parametric inference. The *nonparametric bootstrap* relies on the consideration of the discrete empirical distribution generated by a random sample of size n from an unknown distribution F. This empirical distribution $\widehat{F}_n$ assigns equal probability to each sample item. In *the parametric bootstrap* setting, we consider F to be a member of some prescribed parametric family and obtain $\widehat{F}_n$ by estimating the family parameter(s) from the data. In each case, by generating an iid random sequence, called *a resample* or *pseudo-sequence*, from the distribution $\widehat{F}_n$, or its appropriately smoothed version, we can arrive at new estimates of various parameters or nonparametric characteristics of the original distribution F. This simple idea is at the very root of the bootstrap methodology. In 1981 Bickel and Freedman (1981) formulated conditions for consistency of the bootstrap. This resulted in further extensions of the Efron's methodology to a broad range of standard applications, including quantile processes, multiple regression and stratified sampling. They also argued that the use of bootstrap did not require theoretical derivations such as function derivatives, influence functions, asymptotic variances, the Edgeworth expansion, etc. Singh (1981) made a further point that the bootstrap estimator of the sampling distribution of a given statistic may be more accurate than the traditional normal approximation. In fact, it turns out that for many commonly used statistics the bootstrap is asymptotically equivalent to the one-term Edgeworth expansion estimator, usually having the same convergence rate, which is faster than the normal approximation.

In many recent statistical texts, the bootstrap is recommended for estimating sampling distributions, finding standard errors, and confidence sets. Although the bootstrap methods can be applied to both parametric and non-parametric models, most of the published research in the area has been concerned with the non-parametric case since that is where the most immediate practical gains might be expected. Bootstrap procedures are available in most recent editions of some popular statistical software packages, including SAS and S-Plus. Recently, several excellent books on the subject of bootstrap, resampling and related techniques have become available. Readers interested in gaining some basic background in resampling are referred to Efron and Tibisharani (1993). For a more mathematically advanced treatment of the subject, we recommend Shao and Tu (1995) which also contains a chapter on bootstrapping time series and other dependent data sets.

2.2 Bootstrap Standard Error and Bias Estimates

Arguably, one of the most important practical applications of the bootstrap is in providing conceptually simple estimates of the standard error and bias for a statistic of interest. Let $\hat{\theta}_n$ be a statistic based on the observed sample, arriving from some unknown distribution function F. Assume that $\hat{\theta}_n$ is to estimate some (real-valued) parameter of interest θ, and let us denote its standard error and bias by $se_F(\hat{\theta}_n)$ and $bias_F(\hat{\theta}_n)$. Since the form of the statistic $\hat{\theta}_n$ may be vary complicated the exact formulas for the corresponding bootstrap estimates of standard error ($BESE$) and bias (BEB) may be quite difficult,if not impossible, to derive. Therefore, one usually approximates both these quantities with the help of the multiple resamples. The approximation to the bootstrap estimate of standard error of $\hat{\theta}_n$ suggested by Efron (1979) is given by

$$\widehat{se}_B = \left\{ \sum_{b=1}^{B} [\hat{\theta}^*(b) - \hat{\theta}^*_n(\cdot)]^2/(B-1) \right\}^{1/2} \tag{1}$$

where $\hat{\theta}^*_n(b)$ is the original statistic $\hat{\theta}_n$ calculated from the b-th resample ($b = 1, \dots, B$), $\hat{\theta}^*_n(\cdot) = \sum_{b=1}^{B} \hat{\theta}^*_n(b)/B$, and B is the total number of resamples (each of size n) collected with replacement from the empirical estimate of F (in parametric or non-parametric setting), By the law of large numbers

$$\lim_{B\to\infty} \widehat{se}_B = BESE(\hat{\theta}_n),$$

and for sufficiently large n we expect

$$BESE(\hat{\theta}_n) \approx se_F(\hat{\theta}_n).$$

Similarly, for BEB one can use its approximation $\widehat{bias}_B$ based on B resamples

$$\widehat{bias}_B = \sum_{b=1}^{B} \hat{\theta}_n^*(b)/B - \hat{\theta}_n \tag{2}$$

Let us note that B, total number of resamples, may be taken as large as we wish, since we are in complete control of the resampling process. For instance, it has been shown that for estimating $BESE$, B equal to about 250 typically gives already a satisfactory approximation, whereas for BEB this number may have to be significantly increased in order to reach the desired accuracy (see Efron and Tibisharani 1993 for a discussion of these issues).

2.3 Bootstrap Confidence Intervals

Let us now turn into the problem of using the bootstrap methodology to construct confidence intervals. This area has been a major focus of theoretical work on the bootstrap and in fact the procedure described below is by far not the most efficient one and can be significantly improved in both rate of convergence and accuracy. It is, however, intuitively obvious and easy to justify and seems to be working well enough for the cases considered here. For complete review of available approaches to bootstrap confidence intervals for iid data, see Efron and Tibisharani (1992). As in the case of standard error and bias estimates the methodology for bootstrap interval estimation applies as well to time dependent observations as long as we modify our resampling procedure to ether block or circular bootstrap sampling (see Section 2.4 below).

Let us consider $\hat{\theta}_n^*$, a bootstrap estimate of θ based on a resample of size n (or, in the case of dependent observations, on k blocks of length l, see Section 2.4 below) from the original data sequence $X_1, \ldots, X_n$, and let G_* be its distribution function given the observed series values

$$G_*(x) = Prob\{\hat{\theta}_n^* \leq x | X_1 = x_1, \ldots, X_n = x_n\}. \tag{3}$$

Recall that for any distribution function F and $\tau \in (0, 1)$ we define the τ-th quantile of F (sometimes also called τ-th percentile) as $F^{-1}(\tau) = \inf\{x : F(x) \geq \tau\}$. The *bootstrap percentiles method* gives $G_*^{-1}(\alpha)$ and $G_*^{-1}(1-\alpha)$ as, respectively, lower and upper bounds for $1 - 2\alpha$ confidence interval for $\hat{\theta}_n$. Let us note that for most statistics $\hat{\theta}_n$ the form of a distribution function of the bootstrap estimator $\hat{\theta}_n^*$ is not available. In practice, $G_*^{-1}(\alpha)$ and $G_*^{-1}(1-\alpha)$ are approximated by generating B pseudo-sequences $(X_1^*, \ldots, X_n^*)$, calculating the corresponding values of $\hat{\theta}_n^*(b)$ for $b = 1, \ldots, B$, and then finding the empirical percentiles. In this case the number of resamples B usually needs to be quite large; in most cases it is recommended that $B \geq 1000$.

2.4 Dependent Data

It is not difficult to show that the Efron's bootstrap procedure fails when the observed sample points are not independent. The extension of the bootstrap method to the case of dependent data was first considered by Künsch (1989) who suggested *a moving block bootstrap* procedure which takes into account the dependence structure of the data by resampling blocks of adjacent observations rather than individual data points. The method was shown to work reasonably well but suffered a drawback related to the fact that for fixed block and sample sizes observations in the middle part of the series had typically a greater chance of being selected into a resample than the observations close to the ends. This happens because the first or last block would be often shorter. To remedy this deficiency Politis and Romano (1992) suggested a method based on circular blocks, i.e. on wrapping the observed time series values around the circle and then generating consecutive blocks of bootstrap data from the circle. In the case of the sample mean this method known as *a circular bootstrap* again was shown to accomplish the Edgeworth correction for dependent data.

Let $X_1, \ldots, X_n$ be a (strictly) stationary time series, and as before let $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$ denote a real-valued statistic estimating some unknown parameter interest θ. Given the observations $X_1, \ldots, X_n$ and an integer l $(1 \le l \le n)$ we form the l-blocks $B_t = (X_t, \ldots, X_{t+l-1})$ for $t = 1, \ldots, n-l+1$. In moving blocks procedure of Künsch (1989) the bootstrap data (pseudo-time series) $(X_1^*, \ldots, X_n^*)$ is obtained by generating the resample of k blocks $(k = [n/l])$ and then considering all individual (inside-block) pseudo-values of the resulting sequence $B_1^* \ldots, B_k^*$.

The circular bootstrap approach is somewhat similar except that here we generate a subsample $\tilde{B_1}^* \ldots, \tilde{B_k}^*$ from the l-blocks $\tilde{B_t} = (\tilde{X}_t, \ldots, \tilde{X}_{t+l-1})$ for $t = 1, \ldots, n$ where

$$\tilde{X}_t = \begin{cases} X_t & \text{if } t = 1, \ldots, n, \\ X_{t-n} & t = n+1, \ldots, n+l-1. \end{cases}$$

In either case, the bootstrap version of $\hat{\theta}_n$ is $\hat{\theta}_n^* = \hat{\theta}_n(X_1^*, \ldots, X_n^*)$.

The idea behind both the block and circular bootstrap is simple: by resampling blocks rather than original observations we preserve the original, short-term dependence structure between the observations although not necessarily the long-term one. For the data series for which the long term dependence is asymptotically negligible is some sense (e.g., α-mixing sequences with appropriately fast mixing rate) the method will produce consistent estimators of the sampling distribution of $\hat{\theta}_n - \theta$ as long as $l, n \to \infty$ at the appropriate rate (see e.g., Shao and Tu Chapter 9 for the discussion of the appropriate assumptions). In particular, it follows that the formulas for the approximate $BESE$ and BEB given by (1) and (2), respectively as well as the method of percentiles for confidence estimation are still valid with the block and circular bootstrap. Let us also note that for $l = 1$ both methods reduce to the Efron's bootstrap procedure for iid random variables.

In the next part of the paper we demonstrate the use of the above bootstrap techniques in both parametric and non-parametric setting.

3 Modeling US Mortality Tables

In this section we consider an application of the bootstrap techniques to estimating the variability of the parameters of a general mortality law proposed by Carriere (1992).

3.1 Carriere Mortality Law

The research into a law of mortality has had a long history going back the famous formula of De Moivre. Probably the most popular parametric model of mortality (proposed by Benjamin Gompertz) has been the one that models the force of mortality

$$\mu_x = Bc^x \tag{4}$$

as an exponential function. Whereas Gompertz's model typically fits observed mortality rates fairly well at the adult ages it exhibits some deficiency in modeling the early ones. This is mostly due to the fact that in many populations the exponential growth in mortality at adult ages is preceded by a sharp mortality fall in early childhood and the hump at about age 23 (cf. e.g., Tenenbein and Vanderhoof 1980). This early years behavior of the mortality force is not easily modeled with the formula (4). There has been many attempts in the actuarial literature to modify Gompertz's law in order to obtain a better approximation for the early mortality patterns. It has been observed in Heligman and Pollard (1980) that by adding additional components to Gompertz model one can obtain a mortality law that fits early mortality patterns much better. This idea has been further extended by Carriere 1992 [referred to in the sequel as CR] who has suggested using a mixture of the extreme-value distributions: Gompertz, Inverse Gompertz, Weibull and Inverse Weibull, to model the mortality curve of a population. Namely, he suggested a mortality law based on a survival function $s(x)$ given as the mixture

$$s(x) = \sum_{i=1}^{4} \psi_i s_i(x) \tag{5}$$

where $0 \leq \psi_i$ for $i = 1 \ldots, 4$, $\sum_{i=1}^{4} \psi_i = 1$ are the mixture coefficients and $s_i(x)$ for $i = 1, \ldots 4$ are survival functions based on the Gompertz, Inverse Gompertz, Weibull, and Inverse Weibull distribution functions, respectively, with the corresponding location and scale parameters m_i and σ_i. More precisely,

$$s_1(x) = \exp(e^{-m_1/\sigma_1} - e^{(x-m_1)/\sigma_1}) \tag{6a}$$

$$s_2(x) = \frac{1 - \exp(-e^{-(x-m_2)/\sigma_2})}{1 - \exp(-e_2^m/\sigma_2)} \tag{6b}$$

$$s_3(x) = \exp(-(x/m_3)^{m_3/\sigma_3}) \tag{6c}$$

$$s_4(x) = 1 - \exp(-(x/m_4)^{-m_4/\sigma_4}) \tag{6d}$$

In Carriere mortality model the Gompertz (6a) and Inverse Gompertz (6b) curves, which both are derived from (4), are used to model the mortality of adulthood, whereas the Weibull and Inverse Weibull curves are used to model the mortality of childhood and early adolescence. The introduction of these last two survival functions allows in particular to fit correctly the early years decrease in mortality as well as the hump around age 20, which are both present in many populations mortality patterns. As it has been demonstrated empirically in CR the model (5) fits quite well the patterns of mortality for the entire US population as well as separately US male and female mortality laws. Due to the interpretation of the survival components of the curve (5) Carriere model, unlike that of Heligman and Pollard (1980), gives also the estimates of childhood, adolescence and adulthood mortality by means of the mixture coefficients ψ_i. In the next section we discuss a way of arriving at these estimates.

3.2 Fitting the Mortality Curve

As an example, we fit the general mortality model (5) to the Aggregate 1990 US Male Lives Mortality Table (US Health Dept. 1990). This particular table is of special actuarial interest because it was a data source in developing 1994 GAM and 1994 UP tables. In order to fit the model (5) to the observed mortality pattern, some fitting criterion (a loss function) is needed. In his paper CR has considered several different loss functions but here for the sake of simplicity we will restrict ourselves to only one of them given by

$$\sum_{x=0}^{100} \left(1 - \frac{\hat{q}_x}{q_x}\right)^2, \tag{7}$$

Survival component	Location par (m_i)	Scale par (σ_i)	Mixture par (ψ_i)
$s_1(\cdot)$	80.73	11.26	0.944
$s_2(\cdot)$	42.60	14.44	0.020
$s_3(\cdot)$	0.30	1.30	0.013
$s_4(\cdot)$	23.65	7.91	0.022

Table 1: Estimates of the Carriere survival model parameters for 1990 US Male Mortality Tables. $\psi_4 = 1 - \sum_{i=1}^{3} \psi_i$

where q_x is a probability of death within a year for a life aged x, as obtained from the US table and $\hat{q}_x$ is a corresponding probability calculated using the survival function (5). The ages above $x = 100$ were not included in (7) since, similarly to CR, we have found that the mortality pattern for the late ages in the table had been based on Medicare data and hence might be not representative to the mortality experience of the entire US male population. The parameters of the survival functions (6) as well as the corresponding mixing coefficients ψ_i's were calculated by minimizing the loss function (7). The calculations were performed with a help of the SOLVER add-on function in the Microsoft Office 97 Excel software. The estimated values of the parameters m_i, σ_i^2, and ψ_i for $i = 1, \ldots 4$ are given in the Table 1.

As we can see from the table the modes m_i of the survival components in the Gompertz and the Inverse Gompertz case are equal to 80.73 and 42.60 respectively, whereas in the Weibull and the Inverse Weibull case they are equal to 0.3 and 23.65, respectively. This information reveals that, as intended, both Gompertz components model the later age mortality, and Weibull as well as Inverse Weibull components model the mortality of childhood and adolescence. Moreover, since $\psi_1 = 0.94$ it follows that most of the deaths in the considered population are due to the Gompertz component. A plot of the fitted curve of $\ln(\hat{q}_x)$ using the estimated values of the parameters given in Table 1 along with the values of $\ln(q_x)$ for $x = 0, \ldots, 100$ is presented in Figure 1.

As we can see from the plot the fit seems to quite good except perhaps between the ages of 6 and 12. Let us also note the remarkable resemblance of our plot in Figure 1 with that presented in Figure 4 of CR, which was based on the 1980 US Population Mortality Table.

3.3 Statistical properties of the parameter estimates in Carriere mortality model

The statistical properties of estimators of the unknown parameters ψ_i $i = 1 \ldots 3$ and m_i, σ_i $i = 1 \ldots, 4$ obtained by minimizing the function (7) (we refer to them in the sequel as the *Carriere or CR estimators*) are not immediately obvious. For instance, it is not clear whether or not they are consistent and what is their asymptotic efficiency. Since these questions are related to the problem of consistency of the bootstrap estimation, we will discuss them briefly here.

In order to put our discussion in the context of the statistical inference theory we need to note an important difference between the loss function (7) and the usual, weighted least-square loss. Namely, in our setting the US mortality rates q_x for $x = 1, \ldots, 100$ are not independent observations but rather are themselves estimated from the large number n of the deaths observed in the population.[1] Since the q_x's are estimated based on the same number n of the observed deaths, they are negatively correlated and hence the loss function (7) treated as a random function,

[1]Examining the methodology of creating the US table used as an example in this section we found that in fact the q_x's were obtained by interpolation with the help of Beer's 4-th degree oscillatory formula.

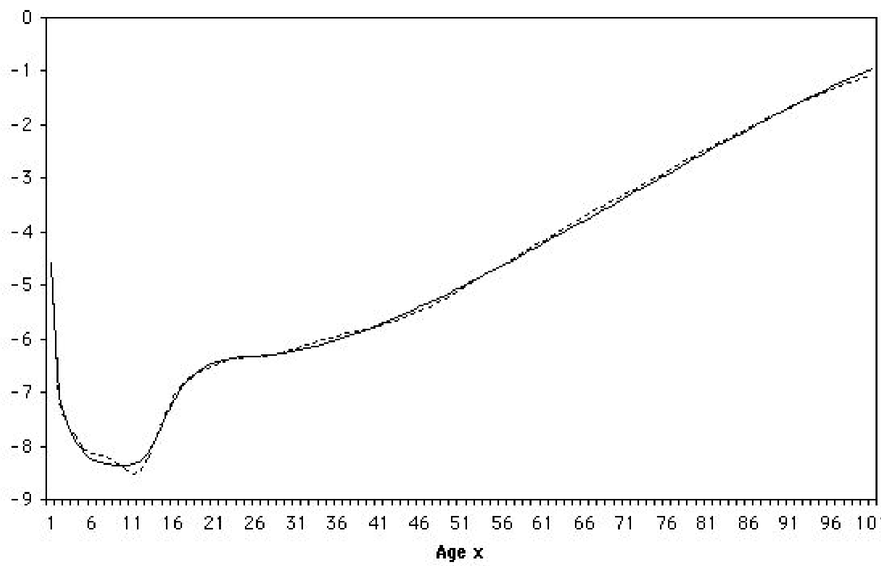

Figure 1: A plot of $\ln(q_x)$ from US Life Table (dashed line) and of $\ln(\hat{q}_x)$ using formulas (6) with the parameters values given in Table 1.

is not a sum of independent identically distributed random variables. In view of the above, the usual statistical methodology of M-estimation does not apply and in order to obtain asymptotics for the CR estimators we need to develop a different approach which is briefly sketched below.

Under the assumption that the true mortality curve of the population indeed follows model (5) with some unknown parameters ψ_i^0, $(i = 1, \ldots, 3)$ and m_i^0, σ_i^0 $(i = 1, \ldots 4)$ one may consider CR estimators associated with the i-th survival component $(i = 1, \ldots, 4)$ as functions of the vector of mortality rates q_x for $x = 1, \ldots, 100$ and expand them around the true parameters ψ_i^0, m_i^0, σ_i^0 using the multivariate Taylor's Theorem. The expansion is valid in view of the Implicit Function Theorem which guarantees that all the local minimizers of (7) are locally continuously differentiable functions of the q_x's as long as the first derivatives of $\hat{q}_x = \hat{q}_x(m_i, \sigma_i, \psi_i)$ are locally continuous and do not vanish (that is, the appropriate Jacobian function is no equal to zero, at least for one age x) around the point ψ_i^0, m_i^0, σ_i^0. This method reveals that CR estimates are consistent and that their join distribution is asymptotically multivariate normal with some covariance matrix of the entries depending upon the values of the true parameters ψ_i^0, $(i = 1, \ldots, 3)$, m_i^0, σ_i^0 $(i = 1, \ldots 4)$ and two first derivatives of the $\hat{q}_x$'s with respect to ψ_i, m_i, σ_i.

3.4 Assessment of the Model Accuracy with Parametric Bootstrap

One way of assessing the variability of CR estimates would be to use the normal approximation outlined in the previous section. However, since the asymptotic variance of this approximation depends upon the unknown parameters which would have to be estimated from the data, the derivation of the exact formula for the covariance matrix as well as the assessment of the performance of variance estimators would be required. In addition, the normal approximation would not account for the possible skewness of the marginal distributions of ψ_i, m_i, and σ_i.. Therefore, as an alternative to this traditional approach, we propose to apply the bootstrap method instead. Since the bootstrap typically corrects for the skewness effect we would hope that it might provide in our setting a better approximations than the methods based on the normal theory.

Let us note that the methods of the nonparametric bootstrap cannot be applied here since we do not know the empirical distribution $\widehat{F}_n$ of the observed deaths. However, we do know the *estimated values of the parameters* in the model (5), since these are obtained by minimizing the expression (7) which is concerned only with the quantities q_x. Hence, assuming that the true population death distribution follows the model (5) we may use the values given in Table 1 to obtain our resampling distribution $\widehat{F}_n$.

Once we have identified the resampling distribution, the bootstrap algorithm for obtaining approximate variances, biases, and confidence intervals for CR estimators is quite simple. We use a computer to generate a large (in our example 20,000 points) random sample from $\widehat{F}_n$ and apply interpolation[2] to obtain the sequence of the pseudo mortality rates q^*_x for $x = 1, \ldots, 100$. In order to find the corresponding bootstrap replication ψ^*_i $i = 1 \ldots 3$ and m^*_i, σ^*_i $i = 1 \ldots, 4$ we minimize the loss function (7), where now the q_x's are replaced by the q^*_x's. The above procedure is repeated a large number (B) times and the resulting sequence of bootstrap replications $\psi^*_i(b)$, $m^*_i(b), \sigma^*_i(b)$ for $b = 1, \ldots, B$ is recorded. The plot of a single replication of the log of q^*_x's curve and the corresponding log of $\hat{q}^*_x$'s curve, i.e., log of the fitted survival curve (5) with the parameters ψ^*_i, m^*_i, σ^*_i is presented in Figure 2.

In order to find the corresponding bootstrap estimates of standard error and bias for each one of the 11 CR estimators in Table 1 we use the formulas (1) and (2). The corresponding bootstrap confidence interval are calculated as well, with the help of the bootstrap percentile method.

The general appropriateness of the above outlined bootstrap procedure is not immediately obvious and in fact requires some theoretical justification as it doe not follow immediately from the general theory. Since the mathematical arguments needed to do so get quite technical and are beyond the scope of the present work, let us just only briefly comment that, generally speaking, for the mere consistency of our bootstrap algorithm, the argument similar to that used to argue the consistency of CR estimators may be mimicked. However, in order to show that our bootstrap estimators in fact have an edge over the normal approximation method a more refined argument based on the Edgeworth expansion theory is needed. For some insight into this and related issues see, for instance, Hall (1992).

The final results based on our bootstrap approximations to the standard error, bias and confidence bounds for all 11 CR estimators given in Table 1 are presented in Table 2. As we can see from these results it seems that only the estimates of the Gompertz component (m_1, σ_1, ψ_1) of the mortality model enjoy a reasonable degree of precision in the sense that their standard errors and bias are small (relative to the corresponding numerical value), and their confidence intervals short. For all the other component both the corresponding values for standard errors and the lengths of the confidence interval, indicate their apparent lack of precision. In particular, the lower bound of the mixture coefficient ψ_2 corresponding to the Inverse Gompertz component (6b) equals zero, which could indicate that the Inverse Gompertz component is simply not present in the considered mortality pattern. In fact, CR fitting the model (5) to US 1980 Mortality data, which has a similar pattern to the one we have considered here, did take $\psi_2 = 0$. Hence, by providing the 95% CI's for

[2] In order to maximize the efficiency of the procedure one should use here the same method of interpolation as the one used to create original values of the q_x's in the table, however, in our example we simply applied linear interpolation to obtain the values of the pseudo-empirical distribution at the integer ages and then to calculate the q^*_x's

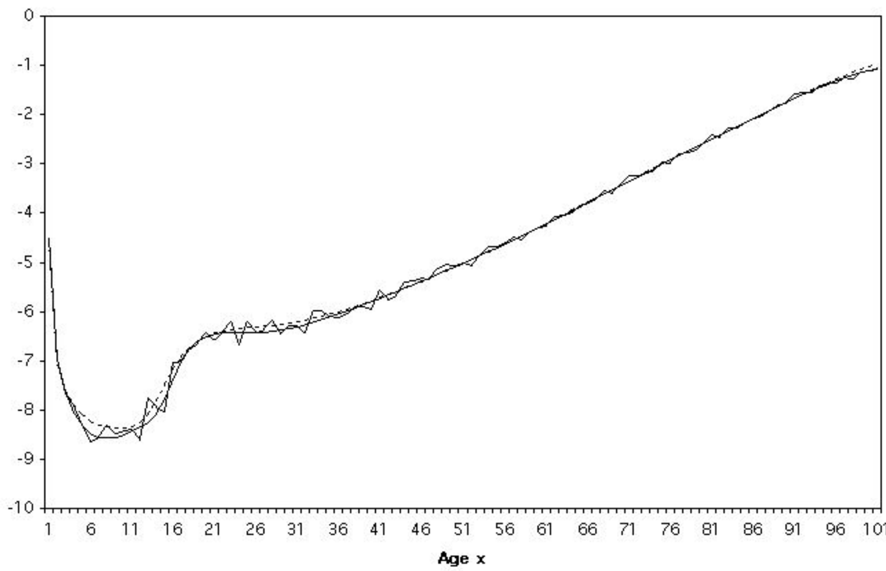

Figure 2: A plot of $\ln(q_x^*)$ (rough solid line) from a single 20,000 point resample of a parametric model (dashed line) and of $\ln(\hat{q}_x^*)$ latest marked by a smooth solid curve). The dashed curve is the original resampling curve with its parameters values given in Table 1.

the estimates of the ψ_i's our bootstrap method also suggests the general form of the mixture in the Carriere law.[3]

For illustration purposes the distribution of $B = 1000$ bootstrap replications for the parameter m_1 along with its 95% confidence interval is presented in Figure 3. Let us note that the histogram of the replication values seems to follow the normal curve (although it is somewhat skewed to the left), which is consistent with our theoretical derivations.

As we have seen above the parametric bootstrap algorithm can be helpful in obtaining a more complete information about the fitted mortality model and its parameters. The drawback of the algorithm presented above is mainly in its very high computational cost, since for every resample of the sequence of 11 values of the parameters we need to first obtain a resample of the fitted mortality rates $\hat{q}_x$, $x = 1 \ldots , 100$. It would be interesting to develop some better methods of bootstrapping in the problems involving the loss functions of type (7), either with the help of the residuals bootstrap method (see Efron and Tibisharani 1993) or maybe some techniques aiming at increasing the efficient of resampling, like e.g. importance method (see Shao and Tu 1995).

[3] Although in this paper we are concerned mostly with the point estimation issues and do not discuss the hypothesis testing, it is not difficult to see that the nonparametric bootstrap method considered here could be also used to perform a formal test of significance of $\psi_i = 0$ (in a similar fashion to the usual test of the coefficients in the regression models). For more on the topic of hypothesis testing with the bootstrap, see e.g., Efron and Tibisharani (1993).

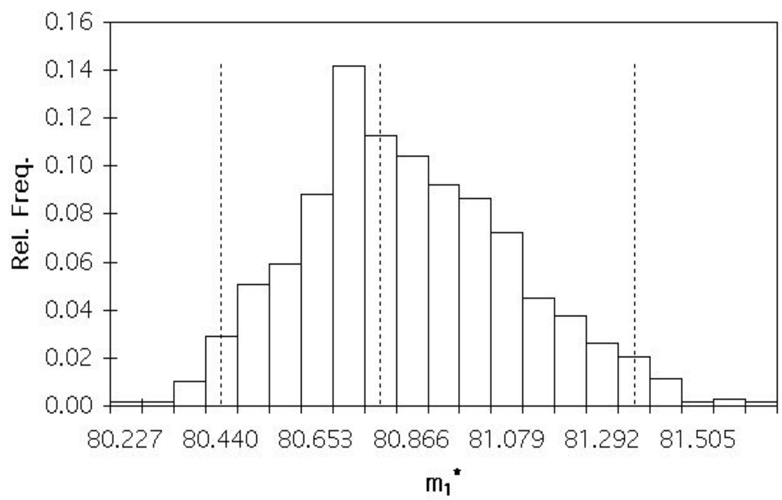

Figure 3: the distribution of $B = 1000$ bootstrap replications for the parameter m_1 along with its 95% confidence interval. The vertical line in the center shown the actual fitted value of m_1 as given in Table 1.

4 Cash flow testing company model

As our second example of the use of the resampling methods, we discuss the applications of the bootstrap to modeling a company cash flow as a function of the interest rates process.

Modern financial methodology of pricing contingent claims, began with the seminal work of Black and Scholes (1973), has had increasing influence on the actuarial practice. The emerging consensus in the actuarial profession calls for consistent use of cash flow testing in analyzing solvency, profitability, fair value of liabilities, and the value of an insurance firm (see, e.g., Vanderhoof and Altman, 1998, also Ostaszewski, 2000). One of the key features of such methodologies is the presence of numerous random variables and stochastic processes in the models, in addition to the standard assumption of randomness of losses or mortality. The most important random effect missing in traditional methodology has been interest rates, or capital asset returns in general. One could also point out randomness of other factors, mainly responses of consumers to economic incentives (lapse rates, growth of new business, mortgage prepayment rates, etc.). However, the effect of interest rates is so pronounced, that the 1991 Standard Valuation Law, and common actuarial practice, now include some form of cash flow testing under varying interest rates scenarios as a standard.

4.1 Interest Rates Process

While there exists a consensus as to the necessity of using interest rates scenarios, and to account for some form of their randomness, one can hardly argue that such a consensus is formed concerning the actual stochastic process governing the interest rates process. Becker (1991) analyzes one common

	m_1	σ_1	ψ_1	m_2	σ_2	ψ_2
Value (from Table 1)	80.73	11.26	0.944	42.60	14.44	0.020
SE	0.245	0.278	0.018	8.257	4.788	0.015
Bias	0.096	-0.120	-0.003	-1.266	-1.653	0.005
Lower Bound 95% CI	80.414	10.521	0.900	26.222	3.007	0.000
Upper Bound 95% CI	81.336	11.608	0.965	54.531	19.932	0.056
	m_3	σ_3	ψ_3	m_4	σ_4	ψ_4
Value (from Table 1)	0.30	1.30	0.013	23.65	7.91	0.022
SE	0.303	2.268	0.001	3.011	3.122	N/A
Bias	0.037	0.254	0.000	-0.179	-0.253	N/A
Lower Bound 95% CI	0.053	0.359	0.011	17.769	3.048	N/A
Upper Bound 95% CI	1.290	10.019	0.015	30.000	13.504	N/A

Table 2: Bootstrap estimates of the standard error, bias and 95% CI for the estimators of the Carriere survival model parameters in 1990 US Male Mortality Tables presented in Table 1. The mixture coefficient ψ_4 is given by $\psi_4 = 1 - \sum_{i=1}^{3} \psi_i$ and hence is not a model parameter.

model derived from the process

$$dI_t = I_t \mu dt + I_t \sigma dW \tag{8}$$

with I_t denoting the interest rate prevailing at time t, W denoting the standard Wiener process, with μ and σ being the drift and volatility parameters. Becker tests the hypothesis, implied by substituting $\mu = 0$ and $\sigma = const$, that the distribution of

$$J_t = \ln\left(\frac{I_{t+1}}{I_t}\right)$$

is normal with mean zero and constant variance. This means that the ratio of interest rates in two consecutive periods has the lognormal probability distribution. However, Becker (1991) rejects the hypothesis of normality and even independence of the J_t's based on the empirical evidence. Of course, (1) is not the only stochastic differential equation proposed to describe interest rate process. Many other processes have been put forth in this very active area of modern mathematical finance. Hull (1997) provides an overview of various stochastic differential equations studied in this context. De La Grandville provides a straightforward argument for approximate lognormality of the variable (1 + rate of return), based on the Central Limit Theorem. But Klein (1993) discusses various reasonings underlying lognormality assumption, and points out that mutual independence of consecutive returns, finite variance of one period return distribution, invariance under addition, and the functional form of the distribution, are very convenient but often unreasonable assumptions. Indeed, many of them have been rejected by a series of recent nonparametric studies of capital markets, discussed by Cerrito, Olson and Ostaszewski (1998). While Klein (1993) argues convincingly against finite variance assumption, the work of Cerrito, Olson and Ostaszewski (1998), among others (notably Fama and French, 1988), aims at the evidence against independence of returns in consecutive periods. Both take away key assumptions of the Central Limit Theorem. Klein (1993) [referenced as KL in the sequel] proposes to replace lognormal distribution by the Stable Paretian, and provides a comprehensive comparison of whole company models under these two hypotheses about interest rates process. As these differing assumptions produce drastically different results based on the same data, KL argues that a valuation actuary must give proper consideration to the possibility that the stochastic dynamics of the interest rate process differ significantly from the simple assumption of lognormality.

4.2 Modeling Interest Rates with Nonparametric Bootstrap of Dependent Data.

Let us note that, unlike in the case of the US Mortality Tables, the historical, empirical realizations of the interest rates processes are typically available, which allows us to use the non-parametric bootstrap methods. In the following sections of this paper we investigate a fully non-parametric model of a company cash flow as a function of the interest rates process $\{J_t\}$. In particular, we do not assume independence of the J_t's, relaxing that assumption to only have interest rates process stationarity. Furthermore, we propose that it may be unreasonable to assume one functional form of the probability distribution. The choice of the non-parametric model here is not merely for the sake of example. In our opinion, this allows to proceed to deeper issues concerning the actual forms of uncertainty underlying the interest rate process, and other variables studied in dynamic financial analysis of an insurance firm. In our view, one cannot justify fitting convenient distributions to data and expect to easily survive the next significant change in the marketplace. What works in practice, but not in theory, may be merely an illusion of applicability provided by powerful tools of modern technology. If one cannot provide a justification for the use of a specific parametric distribution, than a nonparametric alternative should be studied, at least for the purpose of understanding firm's exposures. In this work, we will study the parametric and nonparametric bootstrap methodologies as applied to an integrated company model of KL.

4.3 Assumptions for the Cash-Flow Analysis

While there are many variables involved in a company model, in our example we will limit ourselves to considering a nonparametric estimate of the distribution of 10-th year surplus value as a stochastic function of the underlying interest rates process. We will also assume a simple one-parameter structure of the yield curve. Further work will be needed for extension of this idea to nonparametric estimates of the entire yield curve, and the liability side variables. Stanton (1997) provides one possible nonparametric model of term structure dynamics and the market price of interest rate risk.

In addition to the simple one parameter model of the yield curve, we also make a simplifying assumption of parallel shifts of all interest rates derived from the basic rate (long term Treasury rate). This follows the design of the model company in KL. Clearly, actuarial practice will eventually require greater sophistication of this model. From our perspective, this requires extending bootstrap techniques to several dependent time series (e.g., interest rate series at various maturities). We believe this is definitely an area for important further research.

The model in this work, following that of KL, also assumes a parametric functional form for lapses, withdrawals, and mortality, as well as for prepayments for mortgage securities in the asset portfolio. Again, we believe that further research into nonparametric alternatives to lapse, mortality, and prepayment models, would be very beneficial to our understanding of the probability distribution structure of those phenomena. It should be noted that any research in these areas will require extensive data, which, unlike the interest rates and capital asset returns, is not always easily available.

Overall, we would like to stress that our work in this area has to be, due to numerous limitations, limited, but we do believe that the actuarial profession would greatly benefit from formulation of nonparametric, bootstrap-based models of whole company, incorporating models for all types of uncertainty, i.e., asset returns, lapses, mortality, prepayments, and other random phenomena.

4.4 Company Model Assumptions

Following the design of KL we create a model company for cash flow analysis comparison of the traditional lognormal model of interest rates, and stable Paretian distribution discussed by KL, and our nonparametric model based on resampling. This will allow us to have a direct comparison of parametric and nonparametric models, and make conclusions based on such comparisons.

The company is studied effective December 30, 1990. It offers a single product: deferred annuity. The liability characteristics are as follows:

- Number of policies: 1,000
- Fund Value of Each Policy: 10,000
- Total Reserve: 10,000,000
- Surrender Charge: None
- Minimum Interest Rate Guarantee: 4%
- Reserve Method: Reserve Equals to Account Balance

The assets backing this product are:

- 8,000,000 30-year 9.5% GNMA mortgage pools
- 2,000,000 1 Year Treasury Bills

for a total asset balance of 10,000,000 effectively making initial surplus equal to zero.

Again matching KL analysis, we will assume the following initial yields in effect as of December 31, 1990:

- 1 Year Treasury Bills: 7.00%
- 5 Year Treasury Notes: 7.50%
- 30 Year Treasury Bonds: 8.25%
- Current coupon GNMAs: 9.50%

Treasury yields are stated on a bond-equivalent basis, while GNMAs yields are nominal rates, compounded monthly. The following interest rates are given on 5 Year Treasury Notes at indicated dates:

- December 31, 1989: 7.75%
- December 31, 1988: 9.09%
- December 31, 1987: 8.45%
- December 31, 1986: 6.67%

KL determines the parameters of the lognormal and Stable Paretian distribution for 30 year Treasuries yields based on the historical yields for the period 1953-1976 and 1977-1990. He proceeds then to generate 100 sample 30 year Treasuries yields interest rates scenarios for 10 year period. As in KL, the following assumptions about the model company were made:

- Lapse rate formula (based on competition and credited interest rates, expressed as percentages) is given by

$$q_t^{(w)} = \begin{cases} 0.05 + 0.05\left[100(i_{comp} - i_{cred})\right]^2 & \text{if } i_{comp} - i_{cred} > 0; \\ 0.05 & \text{otherwise;} \end{cases}$$

  with an overall maximum of 0.5.

- Credited interest rate (i_{cred}) is a currently anticipated portfolio yield rate for the coming year on a book basis less 150 basis points. In addition to always crediting no less than the minimum interest rate guarantee, the company will always credit a minimum of $i_{comp} - 2\%$.

- Competition interest rate (i_{comp})is the greater of 1 Year Treasury Bill nominal yield to maturity and the rate 50 basis points below 5 year rolling average of 5 Year Treasury Bond's nominal yield to maturity.

- Expenses and Taxes are ignored.

- There is no annuitization.

- Mortgage prepayment rate ($rate_t$) for the 30 year GNMA pools, calculated separately for each year's purchases of GNMAs:

$$rate_t = \begin{cases} 0.05 + 0.03\left[100(i_{coup,t} - i_{curr})\right] + 0.02\left[100(i_{coup,t} - i_{curr})\right]^2 & \text{if } i_{comp} - i_{cred} > 0; \\ 0.05 & \text{otherwise;} \end{cases}$$

  with an overall maximum of 0.40. Here, $i_{coup,t}$ denotes the coupon on newly issued 30 year GNMAs, issued in year t, and i_{curr} denotes the coupon rate on currently issued 30 year GNMAs. which is assumed to shift in parallel with the yields on 30 year Treasury Bonds. We assume the same, i.e., parallel shifts, about the 1 Year Treasury Bill rates, and the 5 Year Treasury Bond rates.

- Reinvestment strategy is as follows. If the net cash flow in a given year is positive, any loans are paid off first, then 1-year Treasury Bills are purchased until their book value is equal to 20book value of assets, and then newly issued current coupon GNMAs are purchased.

- Loans, if needed, can be obtained at the rate equal to current 1 Year Treasury Bills yield.

- Book Value of GNMAs is equal to the present value of the future principal and interest payments, discounted at the coupon interest rate at which they were purchased, assuming no prepayments.

- Market Value of GNMAs is equal to the present value of the future principal and interest payments, discounted at the current coupon interest rate, and using the prepayment schedule based on the same rate, assuming that the current coupon rate will be in effect for all future periods as well.

- Market Value of Annuities is equal to the account value. All cash flows are assumed to occur at the end of the year. As we had indicated, the projection period is 10 years. We run 100 scenarios and compare them to scenarios run by KL. At the end of 10 years, market values of assets and liabilities are calculated. The distribution of final surplus is then compared.

4.5 Interest Rates Process Assumptions

In order to make a comparison of our non-parametric cash-flow analysis with that presented in KL, we have considered the same set of yield rates on 30-years Treasury bonds from years 1977–1990, namely the average annualized yield to maturity on long-term treasury bonds from years 1977-1990. The data is presented in Table 3 below.

In the first column of Table 3 we give the average yield to maturity on 30-Years Treasury Bonds. These semiannual (bond-equivalent) rates are then converted into effective annual rates I_t, for $t = 1, \ldots 168$ where t is the number of months beyond December 1976. The realizations of these converted rates are presented in column 2. Finally, the realized values of the random time series J_t, calculated according to the formula $J_t = \ln(I_{t+1}/I_t)$ are given in the third column. Let us first note that traditional actuarial assumption that J_t are iid normal do not seem to be satisfied in this case. As shown in Figure 4, the plot of J_t empirical quantiles versus that of a normal random variable indicates that the J_t's possibly comes a distribution other than normal. In his approach KL addressed this problem by considering a different, more general form of a marginal distribution for J_t, namely that of a Stable Paretian family, but he still assumed independence of the J_t's. However, as it was pointed out in the discussion of his paper by Beda Chan there is clear statistical evidence that the J_t's are not, in fact, independent. The simplest illustration of that can be obtained by applying the runs test (as offered for instance by Minitab 8.0 statistical package). There are also more sophisticated general tests for a random walk in a time series data under ARMA(p, q) model as offered by SAS procedure ARIMA (SAS Institute 1996). The results of these tests conducted by us for the data in Table 3 give strong evidence against the hypothesis of independence (both P-values $< .001$).

In view of the above, it seems to be of interest to consider the validity of the stationarity assumption for J_t. Under ARMA(p, q) model again the SAS procedure ARIMA may be used to perform a formal test of non-stationarity either by means of Phillips-Perron or Dickey-Fuller tests. However, a simpler graphical method based on a plot of an autocorrelation function (ACF) and values of the appropriate t-statistics (cf. e.g., Bowerman and O'Connell 1987) may also be used (if we assume that the marginal distribution of J_t has a finite variance). In our analysis we have used both formal testing and graphical methods and found no statistical evidence of non-stationarity of J_t (in conducted tests all P-values $< .001$). The plot of ACF for J_t is given in Figure 2.

In view of the above results, for the purpose of our non-parametric cash-flow analysis we have made the following assumptions about the time series J_t.

(A1) It is a (strictly) stationary time series and

(A2) has the m-dependence structure, i.e., t-th and $s + t + m$-th elements in the series are independent for $s, t = 1, 2, \ldots$ and some fixed integer $m \geq 1$.

The assumption (A1) implies in particular that all the J_t's have the same marginal distribution. Let us note, however, that we have not assumed here anything about the form of this marginal distribution and, in particular, we have made no assumptions regarding the existence of any of its moments. Thus J_t could possibly have an infinite variance or even infinite expectation (which is the case for some distributions from the Stable Paretian family). As far as the assumption (A2)

Year and Month	Yield	Annualized Yield (It)	Jt
1977			
Jan.	7.55	7.69	0.021363
Feb.	7.71	7.86	0.011826
Mar.	7.80	7.95	−0.009187
Apr.	7.73	7.88	0.009187
May.	7.80	7.95	−0.021119
Jun.	7.64	7.79	0.000000
Jul.	7.64	7.79	0.005320
Aug.	7.68	7.83	−0.005320
Sep.	7.64	7.79	0.017191
Oct.	7.77	7.92	0.010440
Nov.	7.85	8.00	0.011620
Dec.	7.94	8.10	0.030367
1978			
Jan.	7.55	7.69	0.021363
Feb.	7.71	7.86	0.011826
Mar.	7.80	7.95	−0.009187
Apr.	7.73	7.88	0.009187
May.	7.80	7.95	−0.021119
Jun.	7.64	7.79	0.000000
Jul.	7.64	7.79	0.005320
Aug.	7.68	7.83	−0.005320
Sep.	7.64	7.79	0.017191
Oct.	7.77	7.92	0.010440
Nov.	7.85	8.00	0.011620
Dec.	7.94	8.10	0.030367
1979			
Jan.	8.94	9.14	0.006836
Feb.	9.00	9.20	0.003401
Mar.	9.03	9.23	0.005644
Apr.	9.08	9.29	0.012311
May.	9.19	9.40	−0.030480
Jun.	8.92	9.12	0.001145
Jul.	8.93	9.13	0.005706
Aug.	8.98	9.18	0.021402
Sep.	9.17	9.38	0.073195
Oct.	9.85	10.09	0.045770
Nov.	10.30	10.57	−0.018069
Dec.	10.12	10.38	0.047510
1980			
Jan.	10.60	10.88	0.138547
Feb.	12.13	12.50	0.017674
Mar.	12.34	12.72	−0.081515
Apr.	11.40	11.72	−0.098192
May.	10.36	10.63	−0.055891
Jun.	9.81	10.05	0.043948
Jul.	10.24	10.50	0.073445
Aug.	11.00	11.30	0.031268
Sep.	11.34	11.66	0.022414
Oct.	11.59	11.93	0.067025
Nov.	12.37	12.75	0.002495
Dec.	12.40	12.78	−0.021821
1981			
Jan.	12.14	12.51	0.054540
Feb.	12.80	13.21	−0.008897
Mar.	12.69	13.09	0.040638
Apr.	13.20	13.64	0.030821
May.	13.60	14.06	−0.049751
Jun.	12.96	13.38	0.048991
Jul.	13.59	14.05	0.043194
Aug.	14.17	14.67	0.035884
Sep.	14.67	15.21	0.000706
Oct.	14.68	15.22	−0.098182
Nov.	13.35	13.80	0.007705
Dec.	13.45	13.90	0.057531
1982			
Jan.	14.22	14.73	0.000000
Feb.	14.22	14.73	−0.051407
Mar.	13.53	13.99	−0.012283
Apr.	13.37	13.82	−0.010085
May.	13.24	13.68	0.051728
Jun.	13.92	14.40	−0.027834
Jul.	13.55	14.01	−0.061176
Aug.	12.77	13.18	−0.058073
Sep.	12.07	12.43	−0.079678
Oct.	11.17	11.48	−0.059587
Nov.	10.54	10.82	0.000000
Dec.	10.54	10.82	0.008722
1983			
Jan.	10.63	10.91	0.023855
Feb.	10.88	11.18	−0.023855
Mar.	10.63	10.91	−0.014577
Apr.	10.48	10.75	0.004881
May.	10.53	10.81	0.038257
Jun.	10.93	11.23	0.043245
Jul.	11.40	11.72	0.037200
Aug.	11.82	12.17	−0.016667
Sep.	11.63	11.97	−0.004430
Oct.	11.58	11.92	0.014987
Nov.	11.75	12.10	0.011319
Dec.	11.88	12.23	−0.011319
1984			
Jan.	11.75	12.10	0.017364
Feb.	11.95	12.31	0.036394
Mar.	12.38	12.76	0.022229
Apr.	12.65	13.05	0.061722
May.	13.43	13.88	0.000769
Jun.	13.44	13.89	−0.017818
Jul.	13.21	13.65	−0.053673
Aug.	12.54	12.93	−0.020744
Sep.	12.29	12.67	−0.026300
Oct.	11.98	12.34	−0.036708
Nov.	11.56	11.89	−0.003563
Dec.	11.52	11.85	−0.006265
1985			
Jan.	11.45	11.78	0.001794
Feb.	11.47	11.80	0.030038
Mar.	11.81	12.16	−0.030038
Apr.	11.47	11.80	−0.038326
May.	11.05	11.36	−0.058271
Jun.	10.44	10.71	0.005877
Jul.	10.50	10.78	0.005844
Aug.	10.56	10.84	0.004845
Sep.	10.61	10.89	−0.010690
Oct.	10.50	10.78	−0.043881
Nov.	10.06	10.31	−0.054343
Dec.	9.54	9.77	−0.015126
1986			
Jan.	9.40	9.62	−0.052442
Feb.	8.93	9.13	−0.117362
Mar.	7.96	8.12	−0.075699
Apr.	7.39	7.53	0.017757
May.	7.52	7.66	0.006750
Jun.	7.57	7.71	−0.041173
Jul.	7.27	7.40	0.008367
Aug.	7.33	7.46	0.039513
Sep.	7.62	7.77	0.010640
Oct.	7.70	7.85	−0.024096
Nov.	7.52	7.66	−0.020517
Dec.	7.37	7.51	0.002759
1987			
Jan.	7.39	7.53	0.020463
Feb.	7.54	7.68	0.001350
Mar.	7.55	7.69	0.090382
Apr.	8.25	8.42	0.063561
May.	8.78	8.97	−0.024723
Jun.	8.57	8.75	0.008306
Jul.	8.64	8.83	0.038290
Aug.	8.97	9.17	0.068350
Sep.	9.59	9.82	0.002132
Oct.	9.61	9.84	−0.072763
Nov.	8.95	9.15	0.019232
Dec.	9.12	9.33	−0.033024
1988			
Jan.	8.83	9.02	−0.047337
Feb.	8.43	8.61	0.023937
Mar.	8.63	8.82	0.037192
Apr.	8.95	9.15	0.031490
May.	9.23	9.44	−0.025797
Jun.	9.00	9.20	0.015778
Jul.	9.14	9.35	0.019942
Aug.	9.32	9.54	−0.028929
Sep.	9.06	9.27	−0.019358
Oct.	8.89	9.09	0.014835
Nov.	9.02	9.22	−0.001134
Dec.	9.01	9.21	−0.007971

Year and Month	Yield	Annualized Yield (It)	Jt	Year and Month	Yield	Annualized Yield (It)	Jt
1989				1990			
Jan.	8.94	9.14	0.007971	Jan.	8.26	8.43	0.029229
Feb.	9.01	9.21	0.017993	Feb.	8.50	8.68	0.007181
Mar.	9.17	9.38	-0.015727	Mar.	8.56	8.74	0.023585
Apr.	9.03	9.23	-0.022886	Apr.	8.76	8.95	-0.003504
May.	8.83	9.02	-0.066891	May.	8.73	8.92	-0.032077
Jun.	8.27	8.44	-0.023708	Jun.	8.46	8.64	0.004815
Jul.	8.08	8.24	0.005036	Jul.	8.50	8.68	0.042361
Aug.	8.12	8.28	0.003761	Aug.	8.86	9.06	0.019421
Sep.	8.15	8.32	-0.018944	Sep.	9.03	9.23	-0.019421
Oct.	8.00	8.16	-0.012824	Oct.	8.86	9.06	-0.037569
Nov.	7.90	8.06	0.000000	Nov.	8.54	8.72	-0.036495
Dec.	7.90	8.06	0.045444	Dec.	8.24	8.41	

Table 3: The average yield to maturity on 30-Years Treasury Bonds along with the annualized yield (I_t) and the natural logarithm of the ratios of consecutive annualized yield rates (J_t)

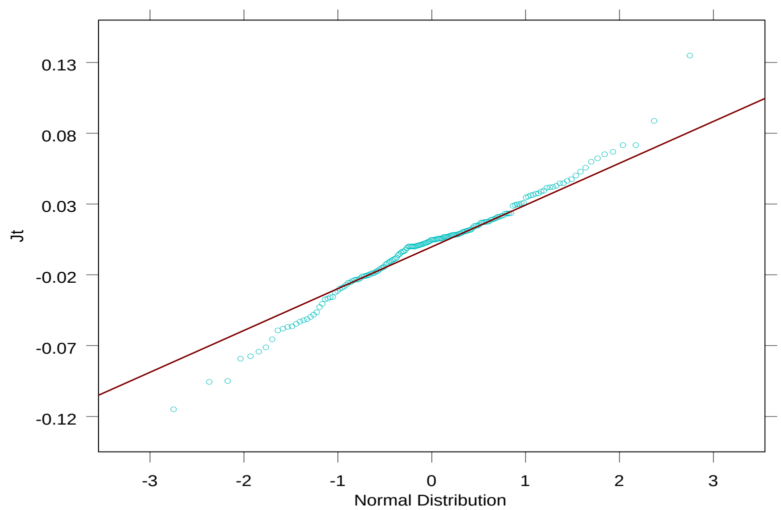

Figure 4: The quantile-to-quantile plot of J_t values given in Table 3 vs the normal distribution

is concerned, the ACF plot in Figure 2 indicates that in the case of long-term Treasury Bonds from years 1953-1976, an appropriate value of m is at least 50. The assumption of m-dependence for J_t seems to be somewhat arbitrary, and in fact from a theoretical viewpoint in order for our bootstrap methodology to apply, the condition (A2) may be indeed substantially weakened by imposing some much more general but more complicated technical conditions on the dependence structure of J_t. These conditions in essence require only that J_t and J_{t+s} are "almost" independent as s increase to infinity at appropriate rate and are known in the literature as *mixing conditions*. Several types of mixing conditions under which our method of analysis is valid are given by Shao and Tue (1995 p.410) For the sake of simplicity and clarity of presentation, however, we have forsaken these technical issues here.

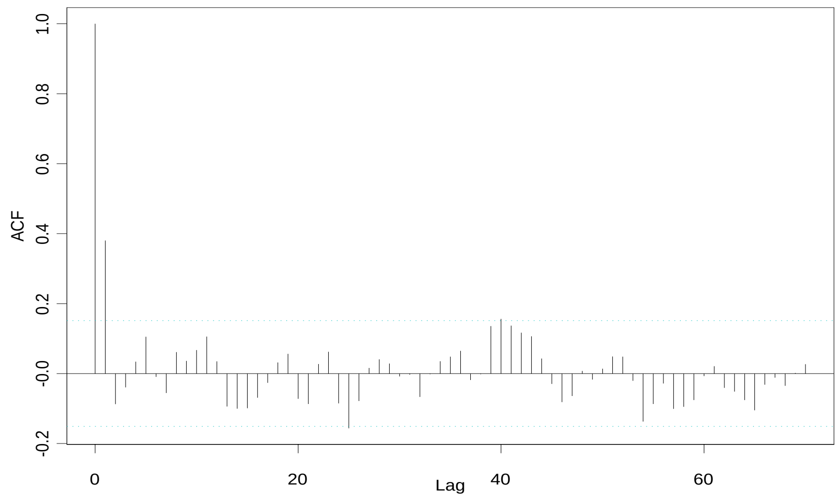

Figure 5: The ACF plot for 16 first lags of the series J_t values given in Table 3

4.6 Bootstrapping the Surplus-Value Empirical Process

In the sequel, let M denote a 10-th years surplus value of our company described in Section 4.4. In general, as described above M is a complicated function of a variety of parameters, some deterministic and some stochastic in nature as well as the investment strategies. However, since in this work we are concerned with modeling cash flow as a function of underlying interest rates process for our purpose here we consider M to be a function of the time series $\{J_t\}$ satisfying assumptions (A1) and (A2) above, assuming all other parameters to be fixed as described in Section 4.4. Under these conditions we may consider M to be a random variable with some unknown probability distribution $H(x) = Prob(M \leq x)$. Our goal is to obtain an estimate of $H(x)$. In his paper KL fitted first the prescribed form of a probability distribution into the interest rates process and then used that model to approximate $H(x)$ by generating interest rates scenarios via Monte-Carlo simulations. In contrast with his approach we shall proceed directly to the modeling of $H(x)$ by bootstrapping the underlying empirical process.

Let us consider a set of $n = 167$ values of J_t process obtained from annualized 30-year Treasury bond yield rates as given in Table 3. If we make an assumption that for the next 10 years the future values of increments of $\ln I_t$ will follow the same stationary process J_t, we may than consider calculating the values of M based on these available 167 data values. In fact, since the J_t's in Table 3 are based on the monthly rates, every sequence of 120 consecutive values of that series will give us a distinct value of M. This process results in $n - 120 + 1 = 46$ empirical values of M. However, by adopting the "circular series" idea, i.e., by wrapping the 167 values from Table 3 around the circle and then evaluating M for all sequences of 120 consecutive J_t we obtain additional 119 empirical values of M and thus a total of $n = 167$ empirical values of M is available. The distribution of these values is presented in Figure 6. However, let us note that the obtained empirical M's are

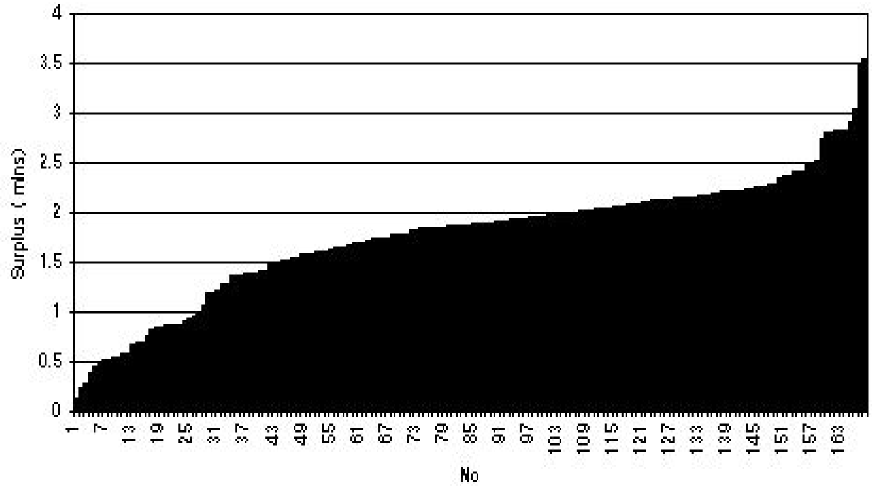

Figure 6: 168 ordered values of the 10-th year surplus, each based on the 120 consecutive values of J_t series given in Table 3 using circular wrapping so that $J_1 = J_{167}, J_2 = J_{168}$, etc.

not independent but rather form a time series which inherits the properties (A1) and (A2) of the original time series J_t. [4] A natural approximation for $H(x)$ is now an empirical process $\widehat{H}_n(x)$ based on the values of M obtained above, say, $m_1, \dots m_n$ (where in our case we would have $n = 167$), defined as

$$\widehat{H}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(m_i \leq x) \tag{9}$$

where $I(m_i \leq x) = 1$ when $m_i \leq x$ and 0 otherwise. By a standard result from the probability theory, under the assumptions that the m_i's are realizations for a time series satisfying (A1) and (A2) we have that for any x

$$\widehat{H}_n(x) \to H(x) \qquad \text{as } n \to \infty.$$

As we can see from the above considerations the empirical process $\widehat{H}(\cdot)$ could be used to approximate the distribution of random variable M and, in fact, a construction of such an estimator would typically be a first step in any non-parametric analysis of this type. In our case, however, the estimate based on the empirical values of the m_i's suffers a drawback, namely that it uses only the available empirical values of surplus value, which in turn rely on very long realizations of the values of the time series J_t, while in fact explicitly using only very few of them. Indeed, for each m_i we need to have 120 consecutive values of J_t, but in the actual calculation of surplus value we

[4] This is not quite true, as there is perhaps a small contamination here due to the use of "wrapping" procedure for calculating empirical values of M. However, it can be shown that the effect of a circular wrapping quickly disappears as the number of available $J_t's$ grows large. For the purpose of our analysis we have assumed the wrapping effect to be negligible.

use only 10 of them. This deficiency of $\widehat{H}_n(\cdot)$ may be especially apparent for small and moderate values of m, i.e., when we deal only with a relatively short dependence structure of the underlying J_t process (and this seems to be the case, for instance, with our 30 year Treasury yields data in Table 3).

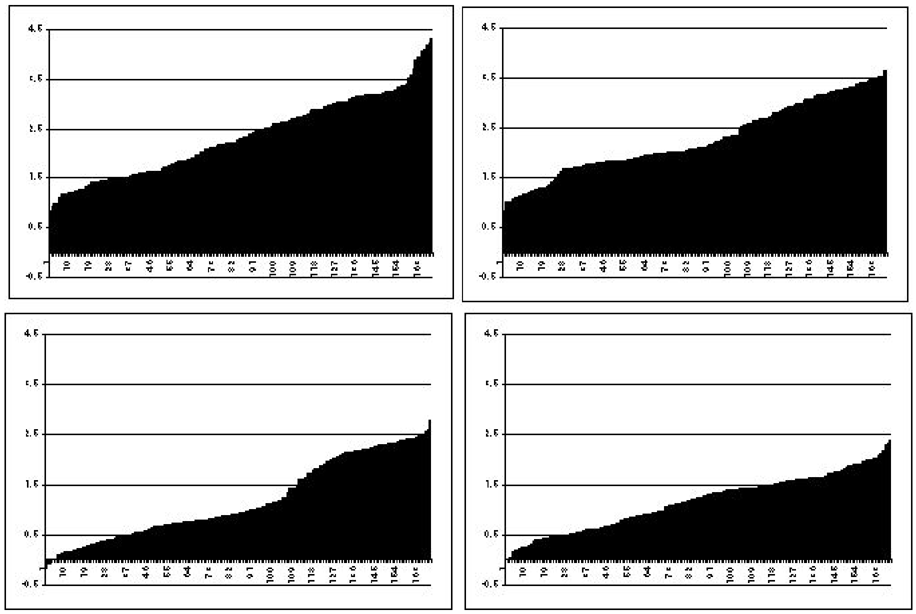

Figure 7: The first four out of of 1000 generated distributions the bootstrapped 10-th year surplus value M^*.

In view of the above, it seems reasonable to consider a bootstrapped version of the empirical process (9), say $\widehat{H}_n^*$, treated as a function of the underlying process J_t with some appropriately chosen block length l. Thus, our bootstrap estimator of $H(x)$ ant any fixed point x (i.e., $\hat{\theta}_n^*$ in our notation of Section 2.4, with $\widehat{H}_n$ being $\hat{\theta}_n$) is

$$\widehat{H}_n^*(x) = \frac{1}{B}\sum_{b=1}^{B} \widehat{H}_{(b)}^*(x) \tag{10}$$

where $\widehat{H}_{(b)}^*(x)$ is a value of $\widehat{H}(x)$ calculated for the particular realization of the pseudo-values $J_1^*, \dots, J_n^*$ generated via circular bootstrap procedure using k blocks of length l as described in Section 2.4, and B is the number of generated pseudo-sequences. [5] Let us note that the above estimate may be also viewed as a function of bootstrapped surplus values of M^*, say, $m_1^*, \dots, m_n^*$ obtained by evaluating the surplus value for pseudo-values of the series $J_1^*, \dots, J_n^*$. For the purpose of the interval estimation of $H(x)$ we consider bootstrap confidence intervals based on the method of percentiles, as described in Section 2.3 where now the bootstrap distribution G_* denotes the distribution of $\widehat{H}_n^*(x)$ at any fixed point x, given the observed values of $J_1, \dots, J_n$.

[5]In order to achieve a desired accuracy of the bootstrap estimator for m-dependent data,the number of generated resamples is usually taken to be fairly large (typically, at least 1000). Of course, the number of all possible *distinct* resamples is limited by the number of available distinct series values (n) and a block length (l).

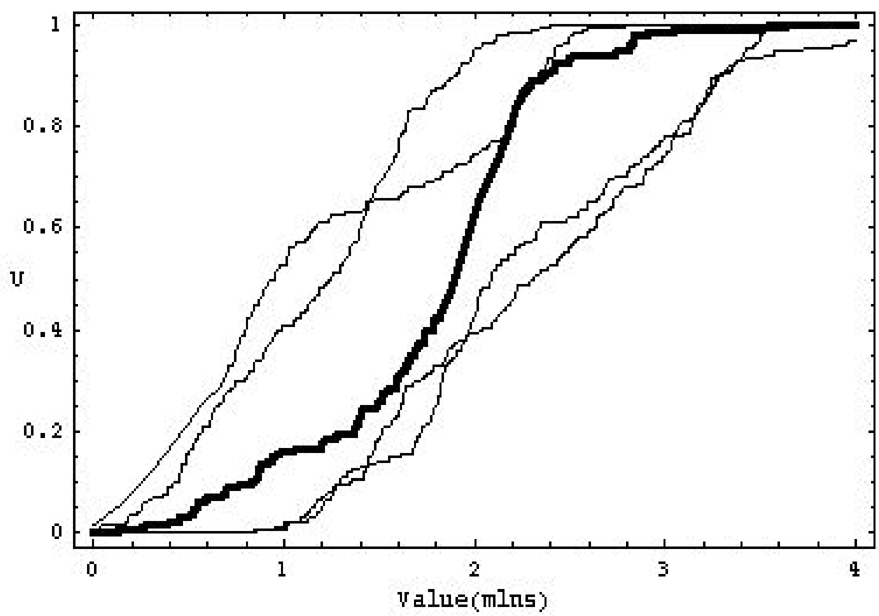

Figure 8: Empirical surplus value process $\widehat{H}(x)$ (thick line) and the four realizations of the bootstrapped surplus value process $\widehat{H}^*_n(x)$ calculated for the distributions given in Figure 7. The values of x are given in millions.

4.7 Distribution of the 10–th Year Surplus. Comparison with the Parametric Models

In this section we shall discuss the results of our non-parametric cash-flow analysis as well as and give their brief comparison with the results obtained in KL. Using the cash flow model outlined above in Section 4.3 we analyze distribution of the 10-th year surplus using the data for average annualized yields to maturity for 30 years Treasury Bonds (as in KL, see Table 3). We have also presented the results of a similar analysis based on a larger set of data, namely on the average yield to maturity on long-term Treasury Bonds for the period 1953-1976.

4.7.1 Bootstrap Estimates of the Surplus Cumulative Distribution

The bootstrap estimates $\widehat{H}^*_n(\cdot)$ of the 10-th year surplus values distribution $H(x) = Prob(M \leq x)$ for a grid of x values ranging between (in millions) along with their 95% confidence intervals were calculated using a method of circular bootstrap as described in Section 2.4 with a circular block length of six months (i.e., in notation of Section 2.4 $l = 6$). Other values of l were also considered and the obtained results were more-less comparable for $1 < l < 6$. In contrast, the results obtained for $l \geq 7$ were quite different, however, they were deemed to be unreliable in view of a relatively small sample size (n=167), since the theoretical result on block bootstrap states that l should be of order smaller than $\sqrt{n}$ in order to guarantee the consistency of bootstrap estimates (cf. Shao and Tu, 1995 Chapter 9). The case $l = 1$ corresponds to an assumption of independence among the J_t's and was not considered here.

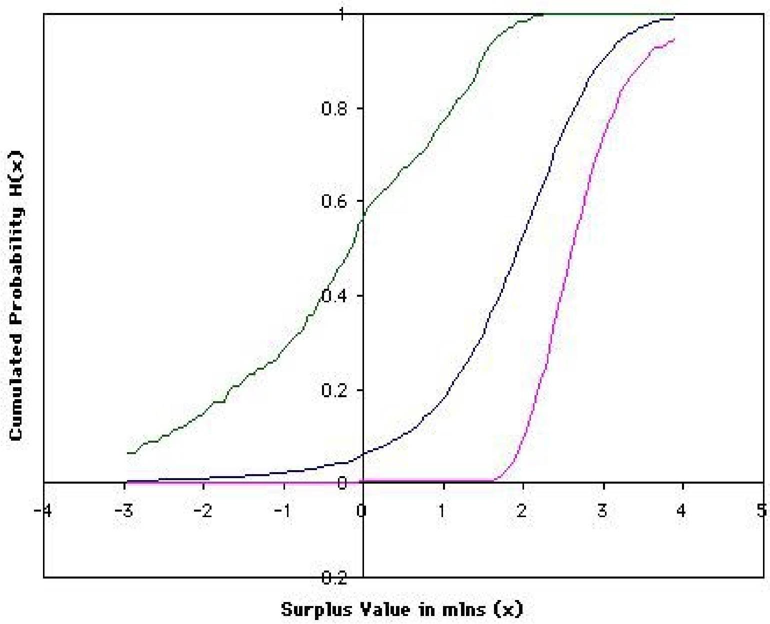

Figure 9: Graph of a bootstrap estimate of the 10-th year surplus distribution $H(x) = Prob(M \leq x)$, along with its 95% confidence bounds, based on the values of the J_t'a calculated from average yields on 30 years Treasury Bonds 1977-1990 given in Table 3. The values of $\widehat{H}_n^*(x)$ were calculated for a very dense grid of x values in order to give an appearance of a continuous curve.

The resulting bootstrap estimator $\widehat{H}_n^*(\cdot)$ of the 10-th year surplus cumulative distribution is presented in Figure 9. A graph of a bootstrap estimate of $H(x) = Prob(M \leq x)$ given by (10), along with the 95% confidence bounds is presented in Figure 9. The values of $\widehat{H}_n^*(x)$ were calculated for a very dense grid of x values in order to give an appearance of a continuous curve. In order to achieve a reasonable accuracy the number of generated resamples for each calculated point x was taken to be 3000; that is, in the formula (10) we had $B = 3000$. The values of the estimators together with their 95% confidence intervals for some of these x values ranging between -2.5 and 4 (in millions of dollars) are also tabulated below in Table 4. Let us note that often for merely practical reasons we would be more interested in estimating $1-H(x)$, i.e., a probability that surplus value will exceed certain threshold, rather than $H(x)$ itself. However, these estimates can be easily obtained from Table 4 as well. For instance, in order to obtain an estimate of positive 10-th year surplus ($Prob(M > 0)$) we subtract the estimate of H(0) given in Table 4 from one. The resulting estimate is then $1 - \widehat{H}_n^*(0) = 1 - 0.06 = 0.94$. In similar fashion the upper and lower confidence bounds for $1 - H(0)$ can be obtained as well. The estimates of probability of exceedance for several different threshold values x are given in Table 5. From the values reported in either table it is easy to see that our bootstrap estimates based on average yields on 30 years Treasury Bonds 1977-1990 are not very precise for x values in the interval from -2 to 2 million, in the sense that they have very wide confidence intervals. For instance, the estimated probability of a negative 10-th year surplus

Surplus Value in mlns (x)	Bootstrap Estimator of H(x)	Bounds for 95% CI	
		Lower	Upper
-2.50	0.01	0.00	0.10
-2.25	0.01	0.00	0.13
-2.00	0.01	0.00	0.15
-1.75	0.01	0.00	0.17
-1.50	0.02	0.00	0.22
-1.25	0.02	0.00	0.24
-1.00	0.02	0.00	0.29
-0.75	0.03	0.00	0.33
-0.50	0.04	0.00	0.40
-0.25	0.05	0.00	0.47
0.00	0.06	0.01	0.57
0.25	0.08	0.01	0.63
0.50	0.10	0.01	0.67
0.75	0.14	0.01	0.71
1.00	0.18	0.01	0.77
1.25	0.24	0.01	0.83
1.50	0.32	0.01	0.91
1.75	0.42	0.02	0.96
2.00	0.53	0.10	0.98
2.25	0.64	0.24	1.00
2.50	0.75	0.42	1.00
2.75	0.84	0.60	1.00
3.00	0.91	0.74	1.00
3.25	0.95	0.84	1.00
3.50	0.97	0.90	1.00
3.75	0.99	0.93	1.00
4.00	1.00	0.96	1.00

Table 4: Bootstrap estimates of $H(x)$ for selected values of x along with their 95% confidence bounds, based on the observed values of the J_t'a calculated from average yields on 30 years Treasury Bonds 1977-1990 as given in Table 3.

x	$1 - \widehat{H}_n^*(x)$	95% CI
0	0.94	(0.43,0.99)
1	0.82	(0.23, 0.99)
2	0.47	(0.02, 0.90)
3	0.09	(0.00, 0.26)

Table 5: Bootstrap estimates of $1 - H(x)$ along with their confidence bounds obtained from the estimates given in Table 4.

equal 0.06 but its interval estimate with 95% confidence is a number between 0.01 and 0.57. Hence the true value of this probability could really be as high as 0.57. It seems that this lack of precision was not the bootstrap's fault, however, but rather an effect of a relatively small sample size for the observed values of the time series. In order to verify whether or not this was truly the case we also have conducted our bootstrap analysis using a different set of data, namely the data on the average yields of the long term Treasury bonds from the period 1953-76.

4.7.2 Estimates Based on the Yields on Long-Term Treasury Bonds for 1953-76

The bootstrap estimate of $H(x)$ based on the data for the average yield to maturity on long-term Treasury bonds for years 1953-1976 for several values of surplus is presented in Table 8. The plot of the estimate along with its 95% confidence bounds is presented in Figure 10. Let us note that in this case the bootstrap estimate is indeed more precise (in the sense of generally shorter confidence intervals) than the one presented in Figure 9 and based on the 30-years Treasury Bond data for 1977-1990, especially in the range between -1 and 1 or so. In particular, the probability of the negative 10-th year surplus based on the data from Table 8 is seen to be estimated as 0.03 with 95% confidence interval between 0 and 0.24. Hence, it seems that using the time series data on the interest rates spanning over 24 years (as opposed to 14 years for 30-years Treasury) we were able to make our bootstrap estimates more precise especially in the middle part of the 10-th year surplus distribution. It would be useful to further investigate that phenomenon by considering even longer periods of time and study the effect of the long time series data sets on the accuracy of our bootstrap estimators. Of course a danger of considering too long data sets is always in the fact that over a very long period of time the key assumptions (A1) and (A2) of Section 4.5 may no longer be valid.

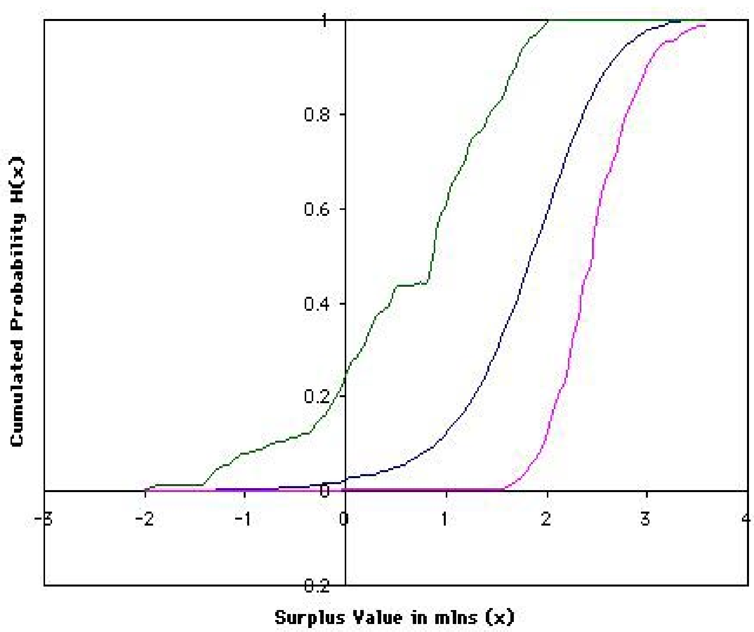

Figure 10: Graph of a bootstrap estimate of H calculated for sufficiently many x values to give it an appearance of a continuous curve, along with its 95% confidence bounds, based on the observed values of the J_t's calculated from average yields on longterm years Treasury bonds 1953-1976 as given in Table 8.

4.7.3 Comparison with the Parametric Analysis Results

Finally, we would like to make a comparison of our results based on non-parametric methodology and assuming only A1 and A2 of Section 4.5 with that obtained under the parametric assumptions about the series J_t considered by KL. In KL an analysis of the 10-th year surplus for a company model described in Section 4.4 was presented under assumptions that the random variables J_t are independent and identically distributed, as well as follow some parametric probability distribution. In his paper KL has considered two competing parametric classes: lognormal and stable Paretian. By estimating the parameters in both models with the help of the data for average yield to maturity

Surplus Value in mlns (x)	Bootstrap Estimator of H(x)	Bounds for 95% CI Lower	Upper
-2.00	0.00	0.00	0.00
-1.75	0.00	0.00	0.01
-1.50	0.00	0.00	0.01
-1.25	0.00	0.00	0.05
-1.00	0.00	0.00	0.08
-0.75	0.01	0.00	0.10
-0.50	0.01	0.00	0.11
-0.25	0.01	0.00	0.16
0.00	0.03	0.00	0.24
0.25	0.04	0.00	0.35
0.50	0.05	0.00	0.43
0.75	0.08	0.00	0.44
1.00	0.12	0.00	0.61
1.25	0.19	0.00	0.74
1.50	0.30	0.00	0.82
1.75	0.44	0.03	0.93
2.00	0.59	0.12	1.00
2.25	0.74	0.32	1.00
2.50	0.86	0.57	1.00
2.75	0.94	0.77	1.00
3.00	0.98	0.90	1.00
3.25	1.00	0.96	1.00
3.50	1.00	0.99	1.00

Table 6: Bootstrap estimates of $H(x)$ for selected values of x along with their 95% confidence bounds, based on the observed values of the J_t'a calculated from the average yields on long-term Treasury bonds 1953-1976 as given in Table 8.

on 30-years Treasury Bonds for years 1977-1990 (our Table 3) and then generating a large number of interest rates scenarios, via Monte Carlo simulations KL has arrived at the empirical estimates of the distributions for the 10-th year surpluses under two different models. For our readers' convenience we have presented the results of his simulations for a set of 100 computer-generated interest rates scenarios in Table 7. As it can be seen from that table, a choice of Stable Paretian model, instead of a usual log-normal one, seems to significantly affect the resulting distribution of surplus. In particular, the probability of negative surplus under a Stable Paretian model seems to be much higher than under a lognormal one. In Figure 11 we present the empirical distributions of the 10-th year surplus based on 100 random scenarios as reported in KL along with our bootstrap estimate. Based on the naive comparison of the three curves it seems that our bootstrap estimate resembles more closely the distribution obtained by KL under the lognormal interest rates model, and in particular, does not indicate the probability of a negative surplus to be as high as indicated under the stable Paretian model for the underlying process of interest rates. However, keeping in mind the confidence bounds on our bootstrap estimate presented in Figure 9 one would have to be very cautious about making such a such statement. In fact, since KL in his method of approximating the true distribution of the 10-th year surplus did not consider any measure of error of his estimators, a simple comparison of the point values on the three curves presented in Figure 11 may be quite misleading. It would be certainly more beneficial for the sake of a validity of our comparisons if some form of an error estimate is produced for the distributions of surplus obtained by KL. This could be done for instance by means of the parametric bootstrap of Section 3.4 which would use the KL's models with his estimated parameters in order to generate bootstrap realizations of $H(x)$.

5 Discussion

The advantages of our non-parametric analysis presented here are quite profound. In the study of Carriere mortality law, bootstrap provides a more complete information about the fitted model

	Surplus Value			Surplus Value	
No	Lognormal	Stable Paretian	No	Lognormal	Stable Paretian
1	(3,935,523)	(121,933,000)	51	1,914,971	1,457,943
2	(1,669,056)	(53,000,540)	52	1,917,268	1,470,926
3	(1,573,561)	(38,609,200)	53	1,928,550	1,496,642
4	(820,183)	(10,555,610)	54	1,983,319	1,499,971
5	(353,858)	(6,178,578)	55	2,005,929	1,516,970
6	(221,643)	(5,981,680)	56	2,014,959	1,583,856
7	(169,454)	(4,820,628)	57	2,048,876	1,589,302
8	(133,671)	(4,546,422)	58	2,073,869	1,635,374
9	(33,429)	(3,875,923)	59	2,103,727	1,638,909
10	19,452	(3,552,448)	60	2,152,675	1,642,424
11	73,653	(3,313,565)	61	2,175,269	1,681,499
12	81,165	(2,298,297)	62	2,190,039	1,755,726
13	90,034	(2,183,848)	63	2,201,091	1,815,831
14	107,789	(1,553,025)	64	2,203,264	1,885,251
15	273,784	(1,420,309)	65	2,224,343	1,888,703
16	440,493	(1,415,358)	66	2,241,418	1,948,011
17	523,963	(1,347,743)	67	2,242,624	1,954,664
18	544,884	(1,121,014)	68	2,253,579	1,974,142
19	563,234	(1,028,966)	69	2,263,852	1,977,154
20	676,511	(916,352)	70	2,288,596	2,024,769
21	823,056	(561,895)	71	2,355,672	2,031,191
22	833,762	(480,189)	72	2,379,694	2,080,799
23	911,935	(436,869)	73	2,390,921	2,119,110
24	956,295	(344,403)	74	2,394,020	2,119,991
25	957,929	(273,309)	75	2,443,931	2,122,132
26	1,070,222	(105,880)	76	2,480,899	2,136,687
27	1,140,108	(15,508)	77	2,484,730	2,186,615
28	1,259,043	151,809	78	2,492,091	2,282,669
29	1,287,237	196,355	79	2,502,697	2,308,413
30	1,338,988	428,123	80	2,522,071	2,313,259
31	1,346,303	552,660	81	2,548,917	2,339,239
32	1,395,566	559,877	82	2,561,432	2,340,308
33	1,442,120	562,296	83	2,576,288	2,384,956
34	1,478,390	563,053	84	2,609,795	2,388,649
35	1,486,547	648,420	85	2,631,667	2,447,761
36	1,488,913	700,347	86	2,632,096	2,471,754
37	1,496,903	815,332	87	2,685,044	2,500,889
38	1,602,665	871,732	88	2,700,908	2,518,084
39	1,607,273	904,513	89	2,726,955	2,523,765
40	1,634,789	958,118	90	2,750,020	2,558,197
41	1,653,456	993,838	91	2,828,278	2,606,791
42	1,655,185	1,019,117	92	2,835,635	2,618,745
43	1,664,288	1,089,538	93	2,850,040	2,641,468
44	1,762,839	1,113,836	94	2,900,917	2,665,568
45	1,765,942	1,231,521	95	2,941,225	2,812,686
46	1,807,418	1,256,956	96	3,085,131	2,918,104
47	1,813,668	1,321,327	97	3,197,324	3,091,850
48	1,838,789	1,349,667	98	3,215,420	3,232,074
49	1,850,946	1,359,857	99	3,249,634	3,254,004
50	1,886,076	1,376,576	100	3,457,466	3,383,891

Table 7: The ordered values (in dollars) for the 10-th year surplus obtained by KL for 100 interest rates scenarios generated from lognormal and stable Paretian distributions fitted to J_t data values as given in Table 3 and treated as independent realizations of a respective random variable.

and its parameters. The bootstrap method could also be used as a guide in determining the final form of the mortality model, by providing a way for testing the significance of any particular model component. While our methodology is computationally intensive, it if very effective in error estimation and identification of the models components which, due to their lack of precision, could be misleading in practice.

The nonparametric bootstrap technique in the whole company asset-liability model has freed us of the most of the typical assumptions about the interest rates process, such as independence, the existence of variance and other moments, etc. Our key assumptions (e.g., process stationarity) are be quite minimal and fairly reasonable under most circumstances. By allowing for the interdependence structure of interest rates time series to be incorporated into the analysis we were able to gain additional insight and precision in our analysis. In addition, the bootstrap approach allowed us to estimate error of our predictions by calculating the confidence intervals for the final

Year and Month	Yield	Year and Month	Yield	Year and Month	Yield	Year and Month	Yield
1953		1956		1959		1962	
Jan.	2.80	Jan.	2.88	Jan.	3.90	Jan.	4.08
Feb.	2.83	Feb.	2.85	Feb.	3.92	Feb.	4.09
Mar.	2.89	Mar.	2.93	Mar.	3.92	Mar.	4.01
Apr.	2.97	Apr.	3.07	Apr.	4.01	Apr.	3.89
May.	3.12	May.	2.97	May.	4.08	May.	3.88
Jun.	3.13	Jun.	2.93	Jun.	4.09	Jun.	3.90
Jul.	3.04	Jul.	3.00	Jul.	4.11	Jul.	4.02
Aug.	3.05	Aug.	3.17	Aug.	4.10	Aug.	3.97
Sep.	3.01	Sep.	3.21	Sep.	4.26	Sep.	3.94
Oct.	2.87	Oct.	3.20	Oct.	4.10	Oct.	3.89
Nov.	2.86	Nov.	3.30	Nov.	4.12	Nov.	3.87
Dec.	2.79	Dec.	3.40	Dec.	4.27	Dec.	3.87
1954		1957		1960		1963	
Jan.	2.69	Jan.	3.34	Jan.	4.37	Jan.	4.14
Feb.	2.62	Feb.	3.22	Feb.	4.22	Feb.	4.16
Mar.	2.53	Mar.	3.26	Mar.	4.08	Mar.	4.15
Apr.	2.48	Apr.	3.32	Apr.	4.17	Apr.	4.15
May.	2.54	May.	3.40	May.	4.16	May.	4.14
Jun.	2.55	Jun.	3.58	Jun.	3.99	Jun.	4.14
Jul.	2.47	Jul.	3.60	Jul.	3.86	Jul.	4.15
Aug.	2.48	Aug.	3.63	Aug.	3.79	Aug.	4.19
Sep.	2.52	Sep.	3.66	Sep.	3.82	Sep.	4.25
Oct.	2.54	Oct.	3.73	Oct.	3.91	Oct.	4.27
Nov.	2.57	Nov.	3.57	Nov.	3.93	Nov.	4.34
Dec.	2.59	Dec.	3.30	Dec.	3.88	Dec.	4.43
1955		1958		1961		1964	
Jan.	2.68	Jan.	3.24	Jan.	3.89	Jan.	4.15
Feb.	2.77	Feb.	3.26	Feb.	3.81	Feb.	4.14
Mar.	2.78	Mar.	3.25	Mar.	3.78	Mar.	4.18
Apr.	2.82	Apr.	3.12	Apr.	3.80	Apr.	4.20
May.	2.81	May.	3.14	May.	3.73	May.	4.16
Jun.	2.82	Jun.	3.19	Jun.	3.88	Jun.	4.13
Jul.	2.91	Jul.	3.36	Jul.	3.90	Jul.	4.13
Aug.	2.95	Aug.	3.60	Aug.	4.00	Aug.	4.14
Sep.	2.92	Sep.	3.75	Sep.	4.02	Sep.	4.16
Oct.	2.87	Oct.	3.76	Oct.	3.98	Oct.	4.16
Nov.	2.89	Nov.	3.70	Nov.	3.98	Nov.	4.12
Dec.	2.91	Dec.	3.80	Dec.	4.06	Dec.	4.14

Year and Month	Yield	Year and Month	Yield	Year and Month	Yield	Year and Month	Yield
1965		1968		1971		1974	
Jan.	4.14	Jan.	5.18	Jan.	5.91	Jan.	6.56
Feb.	4.16	Feb.	5.16	Feb.	5.84	Feb.	6.54
Mar.	4.15	Mar.	5.39	Mar.	5.71	Mar.	6.81
Apr.	4.15	Apr.	5.28	Apr.	5.75	Apr.	7.04
May.	4.14	May.	5.40	May.	5.96	May.	7.07
Jun.	4.14	Jun.	5.23	Jun.	5.94	Jun.	7.03
Jul.	4.15	Jul.	5.09	Jul.	5.91	Jul.	7.18
Aug.	4.19	Aug.	5.04	Aug.	5.78	Aug.	7.33
Sep.	4.25	Sep.	5.09	Sep.	5.56	Sep.	7.30
Oct.	4.27	Oct.	5.24	Oct.	5.46	Oct.	7.22
Nov.	4.34	Nov.	5.36	Nov.	5.44	Nov.	6.93
Dec.	4.43	Dec.	5.65	Dec.	5.62	Dec.	6.78
1966		1969		1972		1975	
Jan.	4.43	Jan.	5.74	Jan.	5.62	Jan.	6.68
Feb.	4.61	Feb.	5.86	Feb.	5.67	Feb.	6.61
Mar.	4.63	Mar.	6.05	Mar.	5.66	Mar.	6.73
Apr.	4.55	Apr.	5.84	Apr.	5.74	Apr.	7.03
May.	4.57	May.	5.85	May.	5.64	May.	6.99
Jun.	4.63	Jun.	6.06	Jun.	5.59	Jun.	6.86
Jul.	4.74	Jul.	6.07	Jul.	5.57	Jul.	6.89
Aug.	4.80	Aug.	6.02	Aug.	5.54	Aug.	7.06
Sep.	4.79	Sep.	6.32	Sep.	5.70	Sep.	7.29
Oct.	4.70	Oct.	6.27	Oct.	5.69	Oct.	7.29
Nov.	4.74	Nov.	6.51	Nov.	5.50	Nov.	7.21
Dec.	4.65	Dec.	6.81	Dec.	5.63	Dec.	7.17
1967		1970		1973		1976	
Jan.	4.40	Jan.	6.86	Jan.	5.94	Jan.	6.94
Feb.	4.47	Feb.	6.44	Feb.	6.14	Feb.	6.92
Mar.	4.45	Mar.	6.39	Mar.	6.20	Mar.	6.87
Apr.	4.51	Apr.	6.53	Apr.	6.11	Apr.	6.73
May.	4.76	May.	6.94	May.	6.22	May.	6.99
Jun.	4.86	Jun.	6.99	Jun.	6.32	Jun.	6.92
Jul.	4.86	Jul.	6.57	Jul.	6.53	Jul.	6.85
Aug.	4.95	Aug.	6.75	Aug.	6.81	Aug.	6.79
Sep.	4.99	Sep.	6.63	Sep.	6.42	Sep.	6.70
Oct.	5.18	Oct.	6.59	Oct.	6.26	Oct.	6.65
Nov.	5.44	Nov.	6.24	Nov.	6.31	Nov.	6.62
Dec.	5.36	Dec.	5.97	Dec.	6.35	Dec.	6.39

Table 8: Average yield to maturity on the long-term Treasury Bonds 1954-1976

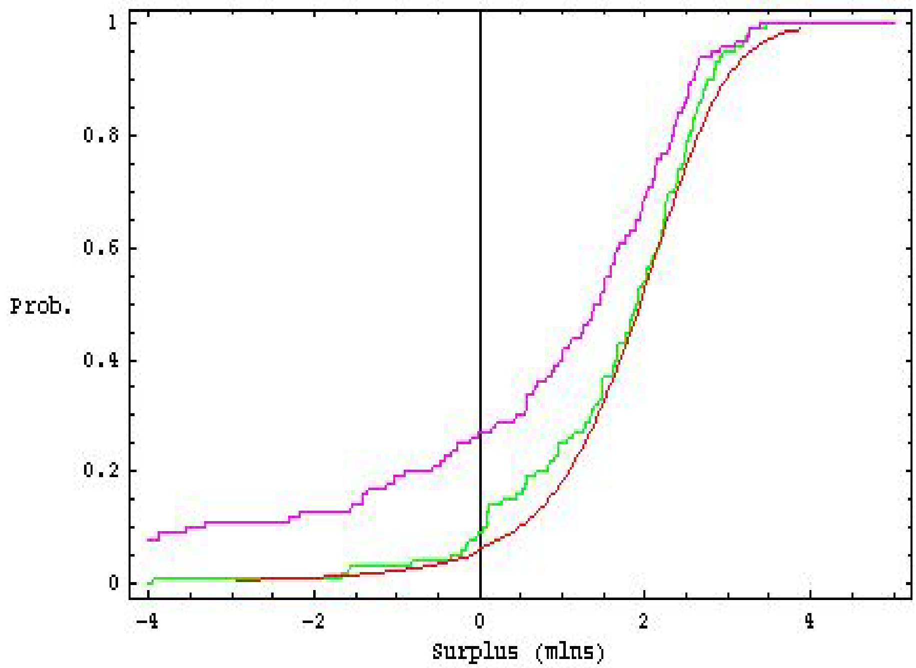

Figure 11: Comparison of our bootstrap estimate of H presented in Figure 9 (lowest curve) with the empirical cumulative distributions functions based on the 100 values of 10-th year surplus obtained by KL and reported in Table 7. Upper curve corresponds to a Stable Paretian model and the middle one to a lognormal one, as described in KL.

non-parametric estimates. This feature of our non-parametric analysis seemed to be particularly relevant in the case of the time series data considered here and in KL, i.e., average yield on 30-years Treasury bonds from 1977 till 1990. In contrast with KL who has not provided in his article any error estimates of his final 10-th year surplus distribution analysis, we were able to determine that the confidence intervals for cash-flow analysis based on this particular data are typically quite wide especially around the middle of the distribution. Thus, it seems that the confidence intervals are simply to wide to make a reasonable prediction based on the available data for 14 years of Treasury Bond yields for the years 1977-1990. We are quite convinced that this area requires further research, and we hope to be a part of it in the future.

It should be also noted that the methodology of this work used for interest rates modeling can, and indeed should, be viewed as a special case of modeling capital markets rates of return. Our approach allows for replacing the interest rates data by, for example, Standard and Poor's 500 Index of stock returns, and can then be directly applied to studying equity assets, or distribution of equities returns in general. As in this work, this would also give better error estimate for the parameters of equities returns. We believe the general area of capital assets returns distribution to hold great promise for bootstrap and other nonparametric estimation.

6 Conclusions

The assessment of the fitted model variability is one of the most important aspects of statistical modeling. In this work we have shown that the bootstrap, in both the parametric and the nonparametric setting, can be quite useful in accomplishing this goal for the actuarial models. In mortality modeling this allows for the better understanding of the fitted model and its sensitivity to the changes in the underlying mortality patterns. The benefits of our approach are even more apparent in cash flow testing model since the stochastic structure there is even richer, and probably still less understood. The remarkable feature of the bootstrap method applied to the interest rates process was that under quite minimal assumptions we were able to obtain the model for cash flow testing which seems to perform equally well, if not better then the parametric ones.

We believe that the present work should be a starting point for further research in the use of bootstrap in actuarial modeling. In particular, mortality estimates can benefit from further, purely nonparametric as well as parametric, bootstrap estimates of bias and standard error applied to other laws of mortality then the one considered here. In the case of cash flow testing, our results are obtained under significant simplifying assumptions, with parametric lapses, mortality, and prepayments. All of these, given appropriately abundant data, can be replaced by nonparametric models. The yield curve structure can be enhanced. The model company should be studied with more complicated and more realistic products. We welcome any contributions in those directions. Of particular significance would be modeling of variable products dependent on equity markets returns. Since equity returns have a much richer, and complicated structure of the distribution of their returns, and equity products may contain very complex options, better understanding of these random phenomena may lead to better product design and risk management.

We firmly believe that bootstrap methodologies deserve a permanent place in the actuarial arsenal. This inclusion can be greatly facilitated by enhanced computing power now available to the profession, as the requirement of such power is truly the only downfall of bootstrap, and indeed other nonparametric approaches. But this is a reasonable price to pay for better understanding and management of risk.

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