

STOCHASTIC ANALYSIS OF AN INSURANCE PORTFOLIO

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Summary

Many papers have appeared recently in the actuarial literature on the problem of studying portfolios of insurance policies when both mortality and interest rates are random. This paper shows how the currently known results for portfolios of identical policies can be extended to portfolios of policies sold to policyholders with different mortality rates (e.g. male - female, smoker - non-smoker) and with different face amounts. Using an Ornstein-Uhlenbeck process for the interest rates, a hypothetical portfolio of temporary insurance policies is used to illustrate the results.

Analyse stochastique d'un portefeuille d'assurances

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Résumé

Bien des publications ont paru récemment dans la littérature actuarielle ayant pour sujet l'étude de contrats d'assurance-vie lorsque la mortalité et les taux d'intérêt sont aléatoires. Le présent article indique comment les résultats actuellement connus pour des portefeuilles de contrats identiques peuvent être extrapolés pour des portefeuilles de contrats dont les assurés présentent des taux de mortalité différents (par exemple, hommes-femmes, fumeurs-non fumeurs), le capital assuré variant lui aussi. En appliquant le processus d'Ornstein-Uhlenbeck pour les taux d'intérêt, un portefeuille hypothétique de contrats d'assurances temporaires est utilisé pour illustrer les résultats.

STOCHASTIC ANALYSIS OF AN INSURANCE PORTFOLIO

1. Introduction.

When trying to determine a contingency reserve or to assess the solvency of a portfolio of life insurance policies, one must consider the random nature of future lifetimes and interest rates.

The interest rates have been treated as random variables by many authors. The references section lists some recent papers on this subject.

This paper builds on the results of Norberg (1993) and Parker (1993a, 1993b). We first derive expressions for the first two moments of the present value of the benefits of a portfolio. We follow the approach used in Parker (1993b) and generalize his results by allowing different types of policies to be included in the portfolio as opposed to considering portfolios of identical policies. It is also slightly more general than the results obtained by Norberg (1993) by allowing, for example, the lives insured to have different mortality rates.

Then the paper looks at a simple example using an Ornstein-Uhlenbeck process for the force of interest. Finally, we use the results of Parker (1993a) to approximate the cumulative distribution of the present value of the benefits of the portfolio chosen for our illustrative purposes.

2. A portfolio of policies.

Consider a portfolio of c temporary insurance policies where each policy is being issued to one of c independent lives. The policies are divided into m groups of policies with similar characteristics.

We will use the following notation:

- c_i : Number of policies in group i . Note that $\sum_{i=1}^m c_i = c$.
- p_i : Proportion of policies in group i , c_i/c .
- n_i : Term of each policy in group i .
- b_i : Benefit (or face amount) payable at the end of the year of death for each policy in group i .
- $K_{i,j}$: Curtate-future-lifetime of the j^{th} ($j=1,2,\dots,c_i$) life insured of group i .
- x_i : Age at issue of each life insured in group i .
- ${}_k|q_{x_i}^{(i)}$: Probability of death in the k^{th} year after issue (at age x_i) using the mortality table appropriate for group i .
- $Z_{i,j}$: Random variable denoting the present value of the benefit that is payable with respect to the j^{th} ($j=1,2,\dots,c_i$) policy of group i .

Then $Z_{i,j}$ may be defined as:

$$Z_{i,j} = \begin{cases} b_i \cdot e^{-y(K_{i,j}+1)} & K_{i,j}=0,1,\dots,n_i-1 \\ 0 & K_{i,j}=n_i,\dots \end{cases} \quad (1)$$

where

$$y(k) = \int_0^k \delta_s \cdot ds \quad (2)$$

i.e. the integral of the force of interest, δ_s .

In order to study $Z_{i,j}$, we need to make the following three assumptions (similar assumptions were made by Frees (1990) and Parker (1993b)):

- A1 - The random variables $\{K_{i,j}\}$ are independent and, for i fixed, they are identically distributed.

A2 - Conditional on knowing the values of $\{y(k)\}_{k=1}^{\omega-x}$, the random variables $\{Z_{i,j}\}$ are independent and, for i fixed they are identically distributed.

A3 - The random variables $\{K_{i,j}\}$ and $\{\delta_{t>0}\}$ are mutually independent.

It is important to note that the random variables $\{Z_{i,j}\}$ are all discounted using the same future forces of interest and therefore, are not independent.

We will use a general approach to study $Z_{i,j}$ in the sense that all we require for now is that the moments of the present value function, $e^{-y(k)}$, and some products of present value functions (such as $e^{-y(k)-y(t)}$) be known.

The m^{th} moment about the origin of $Z_{i,j}$ may be obtained in the following way:

$$E[Z_{i,j}^m] = E[E[Z_{i,j}^m | K_{i,j}]] = \sum_{k=0}^{n_i-1} P(K_{i,j}=k) \cdot (b_i)^m \cdot E[e^{-m \cdot y(k+1)}] \quad (3)$$

where

$$P(K_{i,j}=k) = {}_k|q_{x_i}^{(i)} \quad (4)$$

Let Z be the random variable representing the total present value of all the benefits to be paid with respect to the entire portfolio of c policies. Then

$$Z = \sum_{i=1}^m \sum_{j=1}^{c_i} Z_{i,j} \quad (5)$$

3. Expected value of Z .

The expected value of Z is simply the sum of the expected values of all the $Z_{i,j}$. We then have:

$$E[Z] = E\left[\sum_{i=1}^m \sum_{j=1}^{c_i} Z_{i,j}\right] = \sum_{i=1}^m \sum_{j=1}^{c_i} E[Z_{i,j}] = \sum_{i=1}^m c_i \cdot E[Z_{i,1}] \quad (6)$$

since, by assumption, $E[Z_{i,j}]$ is the same for all $j=1,2,\dots,c_i$.

4. Second moment of Z .

The second moment of Z can be obtained by generalizing the corresponding result for a portfolio of identical policies derived by Parker (1993b, section 4). The result is given in the following theorem.

THEOREM: The second moment of Z under assumptions A1, A2 and A3 is given by:

$$\begin{aligned} E[Z^2] = & \sum_{i=1}^m c_i \cdot E[Z_{i,1}^2] + \sum_{i=1}^m c_i(c_i-1) \cdot E[Z_{i,1} \cdot Z_{i,2}] \\ & + 2 \sum_{i=1}^m \sum_{r=i+1}^m c_i \cdot c_r \cdot E[Z_{i,1} \cdot Z_{r,1}] \end{aligned} \quad (7)$$

where

$$E[Z_{i,1} \cdot Z_{i,2}] = \sum_{k_1=0}^{n_i-1} \sum_{k_2=0}^{n_i-1} b_i^2 \cdot E\left[e^{-y(k_1+1)-y(k_2+1)}\right]_{k_1|q_{x_i}^{(i)}, k_2|q_{x_i}^{(i)}} \quad (8)$$

and

$$E[Z_{i,1} \cdot Z_{r,1}] = \sum_{k_1=0}^{n_i-1} \sum_{k_2=0}^{n_r-1} b_i \cdot b_r \cdot E\left[e^{-y(k_1+1)-y(k_2+1)}\right]_{k_1|q_{x_i}^{(i)}, k_2|q_{x_r}^{(r)}} \quad (9)$$

Proof: To prove (7), we start by expanding Z^2 into a double summation, that is:

$$\begin{aligned} E[Z^2] &= E\left[\left(\sum_{i=1}^m \sum_{j=1}^{c_i} z_{i,j}\right)^2\right] = E\left[\sum_{i=1}^m \sum_{r=1}^m \sum_{j=1}^{c_i} \sum_{s=1}^{c_r} z_{i,j} z_{r,s}\right] \\ &= \sum_{i=1}^m \sum_{r=1}^m \sum_{j=1}^{c_i} \sum_{s=1}^{c_r} E[z_{i,j} z_{r,s}]. \end{aligned} \quad (10)$$

The expected value of $(z_{i,j} \cdot z_{r,s})$ depends on whether the two random variables concern the same life insured (when $i=r$ and $j=s$) or not. We then have:

$$E[Z^2] = \sum_{i=1}^m \sum_{j=1}^{c_i} \sum_{s=1}^{c_i} E[z_{i,j} z_{i,s}] + \sum_{i=1}^m \sum_{r=1}^m \sum_{j=1}^{c_i} \sum_{s=1}^{c_r} E[z_{i,j} z_{r,s}]. \quad (11)$$

The triple summation is given by:

$$\sum_{i=1}^m \sum_{j=1}^{c_i} \sum_{s=1}^{c_i} E[z_{i,j} z_{i,s}] = \sum_{i=1}^m \sum_{j=1}^{c_i} E[z_{i,j}^2] + \sum_{i=1}^m \sum_{j=1}^{c_i} \sum_{s=1, s \neq j}^{c_i} E[z_{i,j} z_{i,s}]. \quad (12)$$

Using A1, A2 and A3, we have, for $j \neq s$,

$$\begin{aligned} E[z_{i,j} z_{i,s}] &= E\left[E[z_{i,j} z_{i,s} | (y(n))_{n=1}^{\omega-x}]\right] \\ &= E\left[E[z_{i,j} | (y(n))_{n=1}^{\omega-x}] \cdot E[z_{i,s} | (y(n))_{n=1}^{\omega-x}]\right] \\ &= E\left[E[z_{i,1} | (y(n))_{n=1}^{\omega-x}] \cdot E[z_{i,2} | (y(n))_{n=1}^{\omega-x}]\right] \\ &= E\left[E[z_{i,1} z_{i,2} | (y(n))_{n=1}^{\omega-x}]\right] = E[z_{i,1} z_{i,2}]. \end{aligned} \quad (13)$$

Similarly, one can show that, for $i \neq r$,

$$E[z_{i,j} z_{r,s}] = E[z_{i,1} z_{r,1}]. \quad (14)$$

and equation (7) immediately follows by substituting (12), (13) and (14) into (11).

Note that

$$\begin{aligned} E[z_{i,1} \cdot z_{i,2}] &= E[b_i^2 \cdot e^{-y(K_{i,1}+1)-y(K_{i,2}+1)}] \\ &= E\left[E\left[b_i^2 \cdot e^{-y(K_{i,1}+1)-y(K_{i,2}+1)} \mid K_{i,1}, K_{i,2}\right]\right]. \end{aligned} \quad (15)$$

Since, by assumption A1, $K_{i,1}$ and $K_{i,2}$ are independent, their joint probability function is the product of their probability functions. Equation (8) therefore follows immediately from equation (15). The proof of (9) is similar. \square

It is of interest to consider the variance of the average cost per policy, z/c , when the number of policies, c , becomes very large while keeping the proportion of policies in each group constant.

It can be shown, under assumptions A1, A2 and A3, that the limiting variance of the average cost per policy as c tends to infinity is:

$$\begin{aligned} \lim_{c \rightarrow \infty} V\left[\frac{z}{c}\right] &= \sum_{i=1}^m p_i^2 \cdot E[z_{i,1} \cdot z_{i,2}] + 2 \sum_{i=1}^m \sum_{r=i+1}^m p_i \cdot p_r \cdot E[z_{i,1} \cdot z_{r,1}] \\ &\quad - \left(\sum_{i=1}^m p_i \cdot E[z_{i,1}]\right)^2. \end{aligned} \quad (16)$$

5. The force of interest.

For our illustrations, we choose to model the force of interest by an Ornstein-Uhlenbeck process. The main results concerning this process are recalled below for completeness. The reader is referred to Parker (1993b, section 6) for more details.

Let δ_t be defined such that:

$$d\delta_t = -\alpha(\delta_t - \delta) \cdot dt + \sigma \cdot dW_t \quad (17)$$

where α , σ and δ are constants with $\alpha \geq 0$ and $\sigma \geq 0$, and W_t is the standard Wiener process (see, for example, Arnold (1974, p.134)).

Note that the parameter α is a friction force bringing the process back towards δ , which is the long term mean of the process. The diffusion coefficient is σ .

Then δ_t is normally distributed with mean:

$$E[\delta_t] = \delta + e^{-\alpha t} \cdot (\delta_0 - \delta) \quad (18)$$

and autocovariance function:

$$\text{cov}(\delta_t, \delta_s) = e^{-\alpha(t+s)} \cdot \frac{\sigma^2}{2\alpha} \cdot (e^{2\alpha s} - 1) \quad s \leq t. \quad (19)$$

Consequently, its variance is:

$$V[\delta_t] = \text{cov}(\delta_t, \delta_t) = \frac{\sigma^2}{2\alpha} \cdot (1 - e^{-2\alpha t}). \quad (20)$$

The function $y(k)$, defined by (2), is an Ornstein-Uhlenbeck position process with mean:

$$E[y(k)] = E\left[\int_0^k \delta_s \, ds\right] = \int_0^k E[\delta_s] \, ds = \delta \cdot k + (\delta_0 - \delta) \cdot \left(\frac{1 - e^{-\alpha k}}{\alpha}\right) \quad (21)$$

and autocovariance function:

$$\text{cov}(y(s), y(k)) = \frac{\sigma^2}{\alpha^2} \min(s, k) + \frac{\sigma^2}{2\alpha^3} \left[-2 + 2e^{-\alpha s} + 2e^{-\alpha k} - e^{-\alpha|k-s|} - e^{-\alpha(k+s)} \right]. \quad (22)$$

So the variance function of $y(k)$ is:

$$V[y(k)] = \frac{\sigma^2}{\alpha^2} k + \frac{\sigma^2}{2\alpha^3} \left[-3 + 4e^{-\alpha k} - e^{-2\alpha k} \right]. \quad (23)$$

Finally, it can be shown that:

$$E[e^{-y(s)-y(k)}] = \exp \left\{ -E[y(s)] - E[y(k)] + \frac{1}{2} \left(V[y(s)] + V[y(k)] + 2 \operatorname{cov}(y(s), y(k)) \right) \right\}. \quad (24)$$

6. Illustrations.

6.1 A Portfolio.

In order to illustrate the results presented in the previous sections, we will use the simple, hypothetical, portfolio described in table 1.

Table 1. Description of the illustrative portfolio.

group i	age x_i	mortality table	face amount b_i ('000)	term n_i	number c_i
1	30	1	50	10	4000
2	35	1	100	5	8500
3	50	1	150	10	6200
4	30	2	50	10	5050
5	40	2	100	10	4800
6	40	3	75	5	2400
7	45	4	25	5	3500

This portfolio of 34450 policies is divided into seven groups with similar characteristics. For example, the 4000 lives insured in group 1 all bought 10-year temporary insurance policies for \$50000, they are all aged 30 and their mortality rates are those of mortality table 1.

The entire portfolio uses four distinct mortality tables. Note that we will use the same mortality table for the first three groups. In our notation, we will then have $q^{(1)} = q^{(2)} = q^{(3)}$. We also have $q^{(4)} = q^{(5)}$. The four mortality tables used here are, in order, the CA 1980-1982 male

ultimate times 1, .9, .8 and .75. One could think of these mortality tables as being male smoker, female smoker, male non-smoker and female non-smoker tables.

6.2 Moments of Z .

The first two moments of Z were calculated with the parameters of the Ornstein-Uhlenbeck process used to model the force of interest arbitrarily chosen to be $\delta=.06$, $\delta_0=.08$, $\alpha=.1$ and $\sigma=.01$.

Some useful intermediary results for the portfolio under consideration are displayed in Table 2. Part A) presents some results for each group taken one at a time. Part B) presents some results about the interaction between the various groups.

Table 2. Summary of intermediary results.

A) Each group.

Group i	$E[z_{i,1}]$	$E[z_{i,1}^2]$	$E[z_{i,1} \cdot z_{i,2}]$
1	.5138	17.7866	.2649
2	.6972	55.7998	.4865
3	9.3730	951.585	88.2003
4	.4627	16.0164	.2148
5	2.1019	141.041	4.4366
6	.6463	38.6321	.4180
7	.3409	6.7891	.1163

B) Interaction between groups, $E[z_{i,1} \cdot z_{r,1}]$.

i	r						
	1	2	3	4	5	6	7
1	—	.3588	4.8336	.2386	1.0841	.3326	.1754
2	.3558	—	6.5453	.3231	1.4679	.4510	.2379
3	4.8336	6.5453	—	4.3529	19.7815	6.0673	3.2002
4	.2386	.3231	4.3529	—	.9763	.2995	.1580
5	1.0841	1.4679	19.7815	.9763	—	1.3607	.7177
6	.3326	.4510	6.0673	.2995	1.3607	—	.2205
7	.1754	.2379	3.2002	.1580	.7177	.2205	—

The values in Table 2 were then used to compute the first two moments of Z using results of sections 3 and 4. Table 3 presents the first two moments of the average cost per policy, Z/c , as well as its standard deviation for our illustrative portfolio. It also presents the corresponding results for other portfolios differing only in size (c).

Table 3. Moments of Z/c for different sizes of portfolio.

c	$E[Z/c]$	$E[(Z/c)^2]$	$sd[Z/c]$
10	2.3589	25.1593	4.4266
100	2.3589	7.5413	1.4060
1000	2.3589	5.7795	.4637
10000	2.3589	5.6033	.1972
34450	2.3589	5.5894	.1580
100000	2.3589	5.5857	.1458
1000000	2.3589	5.5839	.1396
infinity	2.3589	5.5837	.1389

From Table 3, it appears that the average cost per policy of our portfolio of size 34450 should have a distribution relatively close to the limiting one (since the standard deviation of .1580 is relatively close to the limiting value of .1389). This suggests that the mortality risk of the portfolio is small compared to its investment risk. We will use this fact to approximate the distribution of Z in the next section.

6.3 Distribution of Z .

Assuming that the mortality risk is negligible, one is left with the investment risk of the portfolio which can be studied by considering the expected cash flows of the portfolio (see, for example, Frees (1990, proposition 5) and Parker (1992b, section 3). We therefore study the moments and the distribution of the present value of those expected cash flows.

Table 4 presents the expected cash flows for our portfolio. The formula for the expected cash flow at time k , $E[CF_k]$, is:

$$E[CF_k] = \sum_{i=1}^m \sum_{k=0}^{n_i-1} c_i \cdot b_{i,k} | q_{x_i}^{(i)}. \quad (25)$$

Table 4. Expected Cash Flows.

Time	1	2	3	4	5	6	7	8	9	10
expected cash flow	9233	10040	10958	11968	13045	11415	12430	13540	14717	15935

The first two moments about the origin of the present value of these ten cash flows may be obtained from equations (3) and (5) of Parker (1993a) which are:

$$E \left[\sum_{k=1}^{10} E[CF_k] \cdot \exp\{-y(k)\} \right] = \sum_{k=1}^{10} E[CF_k] \cdot E \left[\exp\{-y(k)\} \right] \quad (26)$$

and

$$\begin{aligned} E \left[\left(\sum_{k=1}^{10} E[CF_k] \cdot \exp\{-y(k)\} \right)^2 \right] &= E \left[\sum_{k=1}^{10} \sum_{j=1}^{10} E[CF_k] \cdot E[CF_j] \cdot \exp\{-y(k)-y(j)\} \right] \\ &= \sum_{k=1}^{10} \sum_{j=1}^{10} E[CF_k] \cdot E[CF_j] \cdot E \left[\exp\{-y(k)-y(j)\} \right]. \end{aligned} \quad (27)$$

The expected value is 81264 (as a check, this value is also equal to 34450×2.3589 , see Table 3) and the standard deviation is 4785.

This value of 4785 may also be obtained by multiplying the number of policies by the limiting standard deviation of the average cost per policy, that is 34450×1.389 (see Table 3). Since the exact standard deviation of z for our portfolio is 5443 (34450×1.580), the standard deviation is underestimated by 658 (i.e. $5443 - 4785$) when the expected cash flows are studied instead of the random cash flows.

Using the method described in section 5 of Parker (1993a), the cumulative distribution of the expected cash flows was approximated. Table 5 presents the probability that the present value of the expected cash flows will be smaller than a given value z , $F_z(z)$, for different values of z .

Table 5. Distribution of the present value of the expected cash flows.

z	$F_z(z)$	z	$F_z(z)$
63320.54	0.000006	85650.42	0.805462
65314.28	0.000148	87843.54	0.898161
67308.02	0.000889	90036.65	0.952435
69301.76	0.004249	92229.77	0.978050
71295.50	0.015401	94422.88	0.988969
73289.24	0.044351	96615.99	0.993683
75282.98	0.107483	98809.11	0.995746
77276.72	0.207765	101002.22	0.996914
79270.46	0.345304	103195.34	0.997715
81264.20	0.507288	105388.45	0.998316
83457.31	0.674519	107581.56	0.998882

The values in Table 5 can be used to estimate the contingency margin that should be added to each premium in order for the portfolio to be profitable with a given probability. For example, the net single premium of each policy in our portfolio should be loaded by about $10.8\% \left(\frac{90037}{81264} - 1 \right)$ if we want it to be profitable with a probability of at least .95. (Note that

the 10.8% slightly underestimates the true contingency margin that should be added, because the expected cash flows were used instead of the random cash flows.)

7. Concluding Remarks.

In this paper, we have developed results useful for studying existing insurance portfolios when future lifetimes and interest rates are random. The approach used here is easily adaptable to different situations.

A hypothetical portfolio was analyzed when modeling the force of interest by an Ornstein-Uhlenbeck process. The type of information contained in tables 3 and 5, in particular, should be helpful when pricing and valuing a portfolio. For our illustration, it was found that the total net single premium should be loaded by about 10.8% (or slightly more) to ensure a probability of .95 of making a profit.

Finally, the author is currently using these results to study an existing portfolio of 10-year temporary insurance policies. We hope to present the findings briefly at the Colloquium.

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